Complete integrability in classical gauge theories

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Abstract

We consider completely integrable classical field theory models with a view to identifying the properties which characterize their integrability. In particular, we study the infinite sets of 'hidden' symmetries, and the corresponding transformations carrying representations of infinite dimensional loop algebras, of the following models: the chiral-field equations in two dimensions, the self-dual sector of pure gauge theories in 4 dimensions, the functional (loop-space) formulation of 3-dimensional gauge theories, and some sectors of the extended supersymmetric gauge theories. We also construct an infinite number of conserved spinor currents for the latter theories. The (non-) integrability of the full four dimensional Yang-Mills equations is studied; and a local approximation for the non-integrable phase factor of gauge theories on an arbitrary, infinitesimally small, straight-line path is presented. Finally, we study classical gauge theories in dimensions greater than four; and obtain, in analogy to the self-duality equations, algebraic equations for the field-strength which automatically imply the higher dimensional Yang-Mills equations as a consequence of the Bianchi identities. The most interesting sets of equations found are those in eight dimensions which have a structure related to the algebra of the octonions.

"If thought discovered in the shimmering mirrors of phenomena eternal relations capable of summing them up and summing themselves up in a single principle, then would be seen an intellectual joy of which the myth of the blessed would be but a ridiculous imitation."

— Albert Camus, 'The Myth of Sisyphus'.

Contents

Preface

This thesis represents work carried out in the Department of Mathematical Sciences of the University of Durham between October 1980 and July 1983.

Since October 1981 I have held an ORS (Overseas Research Student) Award from the Committee of Vice Chancellors and Principals of UK Universities.

Nothing in the introductory chapter of this thesis is claimed to be original. However, original work is presented in all subsequent chapters. Much of this work was done in collaboration with others: chapter 2 and section 3.1 (the results of which appeared in Nuclear Physics B194(1982)232), together with section 3.3 and chapter 5 , are based on work done with D.B.Fairlie; chapter 6 is based on work done in collaboration with E.Corrigan, D.B.Fairlie and J.Nuyts and published in Nuclear Physics B214(1983)452. The material of chapter 4 is under preparation for publication.

For much encouragement and many helpful discussions at various stages of this work, I should like to thank Drs. W.J.Zakrzewski and E.Corrigan. I am especially grateful to Dr. D.B.Fairlie, under whose guidance the work presented herein was carried out, for a great deal of help and encouragement. I should also like to thank Dr. R.S.Ward for helpful comments and discussions on the work of chapter 6.

This thesis is dedicated to my parents.

Durham,

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Chapter 1: Introduction.

Non-abelian gauge theories [1] have yielded the most promising description of the empirically observed properties of elementary particles. The weak and electromagnetic interactions, particularly in view of the recent (tentative) discovery of the $W^{\pm}$ and Z^{0} particles, receive a natural explanation if one assumes that they are descibed by a gauge-invariant theory. There also exists much motivation for the current interest in QCD as the theory underlying the strong interactions [2] . However, non-abelian gauge theories have not hitherto yielded themselves to an acceptable quantization scheme. Particularly for the strong-coupling limit of QCD, this has resulted in the impossibility of making quantitative predictions of the theory which could be compared with experiment. Even the observed confinement of colour has not been established as a mathematical property of the QCD lagrangian. This intractability of the theory reflects itself in the hiatus between what we fancy we know and understand (theoretically) about particle interactions and what we actually do know (experimentally). Clearly, if this gulf which separates us from an understanding of elementary particle interactions is to be bridged, reliable methods of quantizing the theory need to be developed.

Pure non-abelian gauge theory is described by the Yang-Mills action $S = -\frac{1}{4g^2}\int d^4x \; F^a_{\mu\nu} F^{\mu\nu a}$, where the integral is over (Minkowski) spacetime, g is the coupling constant of the Yang-Mills field, and $F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g A^b_\mu A^c_\nu f^{abc}$ are components of the Lie algebra-valued curvature $F_{\mu\nu} = F^a_{\mu\nu} T^a = [D_\mu, D_\nu]$, where the gauge-covariant derivative $D_\mu = \partial_\mu + g A^a_\mu T^a$, and T^a are the generators of the Lie algebra with structure constants f^{abc}. The equations of motion for the potential A_μ derived from this action, $D_\mu F_{\mu\nu} = 0$, are nonlinear in A_μ; and it is primarily this nonlinear nature which gives rise to the intractability of the theory. Recent (coupling-constant-based) perturbative schemes for quantizing the theory are not very attractive because in trying to mimic the successful canonical quantization of electrodynamics, where the LSZ formalism may be used to relate asymptotic physical states to the field variables, the nonlinear nature of the theory is effectively ignored. It seems that the notion of quanta, which appears in a non-interacting theory as a property of the Fock space quantization of the free field and in a conventional interacting theory (like QED) through the Feynman-Dyson expansion of green functions and S-matrix elements and the associated idea of the completeness of the asymptotic scattering states, is not appropriate for QCD, since quarks and gluons do not appear as asymptotic particles.

There is thus much motivation for attempting to develop non-perturbative (or even non-coupling constant based perturbative) schemes of quantizing Yang-Mills theory. One recognised method is to use the saddle point approximation of the functional integral in order to covariantly quantize the theory. This method relies heavily on an understanding of the classical field equations , since explicit solutions are a pre-requisite for developing the quantum theory. Although this program motivated much of the early work on instanton solutions to the self-duality equations, it has hitherto been found technically impossible to execute [8] . However, the instanton solutions were found to have a structure remarkably similar to the soliton solutions of two-dimensional theories. Indeed, Polyakov, Belavin and Zakharov, Ward, Yang, and others found many similarities between the self-duality equations and the equations of motion of completely integrable two dimensional theories. This work has raised the possibility of further similarities between nonabelian gauge theories and completely integrable model field theories. The most outstanding result for the latter theories is the quantum spectral transform developed by Faddeev and his collaborators [3-6]. This yields an exact canonical quantization of the theory in an intrinsically nonlinear fashion, incorporating a nonlinear superposition principle in order to build a nonperturbative Fock space. Since this method uses soliton-like structures as the asymptotic physical states, it raises the intriguing possibility that quantum integrability and exact quantizability are inextricably linked.

Although the quantum spectral transform is an intrinsically quantum method (in that it does not depend on a 'background' classical structure), there are many structural similarities between the classical and quantum versions of two dimensional integrable models. It is therefore useful to study classical gauge theories. Not only would classical solutions be useful for a possible covariant quantization scheme, but any structural similarities with 2d completely integrable models would increase the chances of constructing tractable quantum gauge models which take the nonlinear nature of the theory seriously. Many of the two dimensional integrable equations describe actual physical systems [5,6] ; for instance, the Korteweg-de Vries (K-dV) equation, first derived in the study of long water waves in a shallow channel, describes many phenomena in plasma physics; and if there is any lesson to be learnt from the physics of one and two dimensional systems, it is surely that nature seldom forgoes the use of available nonlinearities.

Further, apart from matters concerning quantization, the classical description has a mathematical appeal which in itself justifies further consideration of classical gauge theories. It is the primary motivation of this thesis to investigate the extent to which the classical field equations of non-abelian gauge theories are integrable and to study the integrable sectors of the theory. In particular, we aim to identify and study the properties of these equations which characterize their (possible) integrability. Foremost amongst these properties is the possibility of writing the equations of motion in the Lax form [7] :

$$\partial_0 L = [L, A] \quad , \tag{1}$$

(where L,A are linear differential operators), which we may rewrite as

$$[\partial_0 + A, L] = 0 \quad , \tag{2}$$

which is just the compatibility condition for the set of equations:

$$(\partial_0 + A)\psi = 0 \tag{3a}$$

$$L\,\psi = 0 \quad . \tag{3b}$$

If the dimension of space-time is two, and if $L = \partial_1 + B$, for some B, eq. (2) is then just the condition for the vanishing of the curvature of the connection form C_μ with components $C_0 = A$, $C_1 = B$; i.e. (2) is equivalent to

$$\mathfrak{F}_{\mu\nu} \equiv \partial_\mu C_\nu - \partial_\nu C_\mu + [C_\mu, C_\nu] = 0 \quad . \tag{4}$$

If A and B are two components of a Lorentz vector, then the differential equation implied by (4) will be relativistic; but in general this need not be the case. We may remark that (4) is just a manifestation of the Poincare lemma: $d(dw) = 0$ (for a differential form w), a statement of the equality of mixed second partial derivatives, which is the source of most integrability conditions for partial differential equations.

Once the equation of motion has been cast into the form of (4), something important is guaranteed. For the corresponding linear system not only guarantees (formally) the existance of an infinity of conserved quantities [7a], but also makes the equation of motion amenable to an algebraic method of solution. That it leads to an infinite number of conservation laws may be demonstrated, albeit only for a restricted class of models, using an argument due to Polyakov (see [9,10]) which is particularly instructive for gauge theory type models. This considers a scattering problem

$$(\partial_\mu + C_\mu)\psi = 0 \tag{5}$$

with $\psi(-\infty, t) = 1$, $\dot\psi(+\infty, t) = 0$.

This problem exists (in any dimension) if (4) is satisfied. Now (4) need not necessarily be equivalent to the equations of motion. It could, for instance, be some identity in the problem - like a Bianchi identity. To proceed we need to invent a new combination of C_μ's depending on a parameter λ in such a way that a zero-curvature condition for $C'_\mu(\lambda) \equiv F(C_\mu, \lambda)$ now implies the equations of motion in addition to t he previous identity. Now, identifying $C'_\mu(\lambda)$ with the potential of the scattering problem, it is clear that

$$\frac{dQ}{dt} = 0 \quad ; \quad Q = P\, e^{\int_{-\infty}^{\infty} C_1'(\lambda)\, dx} \quad . \tag{6}$$

Expanding the path-ordered exponential in a power series in yields an infinity of conserved charges. Thus (6) is a compact representation for the generator of these charges. The crucial point to note about Polyakov's argument is that the integrability condition is precisely a statement of the path-independence (i.e. integrability) of the phase factor of parallel transport [12] :

$$\Psi_{x_1,x_2} = P\, e^{\int_{x_1}^{x_2} C'_\mu(\lambda)\, dx_\mu} \quad . \tag{7}$$

Indeed, it is clear that the conservation law (6) stems from the boundary conditions of the scattering problem (5) for which $\Psi_{-\infty,x}$ is clearly a formal solution. We may directly check the path-independence of (7) by considering its variation due to a variation of the path, $x_\mu(t)$, parametrized by t [11]:

$$\frac{\delta}{\delta x_\mu(t)} \Psi_{x_1,x_2} = C'_\mu(x_1)\Psi_{x_1,x_2} - \Psi_{x_1,x_2} C'_\mu(x_2) + \Psi_{x_1,y} \frac{dy_\nu}{dt} \mathfrak{F}_{\mu\nu}(y)\, \Psi_{y,x_2} \quad .$$

The first two terms are contributions of the end-points of the path; and we see that any path dependence (i.e. non-integrability) of the phase factor would be entirely encoded in the curvature $\mathfrak{F}_{\mu\nu}$ of the connection C'_μ (which in the present case is flat).

Infinitely many conservation laws are important because they suggest the possibility that the equations of motion are completely integrable (or perfect) in the sense that the phase space for the system can be reduced to a completely separable one by a canonical transformation to action-angle variables [25]. We recall [14] , that for a hamiltonian system with a finite number, N, of degrees of freedom, the existence of N commuting integrals of motion means, by virtue of Liouville's theorem, that the system is fully integrable, i.e. that it is possible to separate the variables and introduce action-angle variables. Integrable systems are also not completely randomized, since there is no exchange of energy between the degrees of freedom. For infinite-dimensional hamiltonian systems, the existence of an infinity of commuting integrals is only a necessary (but not sufficient) condition of integrability. We note, however, that a canonical transformation to action-angle variables is implicit in the Inverse Scattering transform for such systems [5] , since this incorporates what is effectively a nonlinear mapping to a free-field theory. Similar nonlinear mappings are also the basis of methods which have been found to be useful for the solution of gauge theory-type systems, namely the twistor methods e.g.[17] and the Riemann-Hilbert method [26,27]. Moreover, just as the representation of nonlinear evolution equations, like the KdV equation, as integrability conditions for a system of linear equations made it possible to develop the inverse scattering method [7], these other solution generating techniques also depend on a linear system of the form (5):

$$(d + C(\lambda,x)) = 0 , \tag{8}$$

where d denotes a partial differential operator, possibly depending on . The procedure of e.g. [26] begins with two known solutions (C_0, Ψ_0) and $(C_0, \widetilde{\Psi}_0)$ of (8), where Ψ_0 and $\widetilde{\Psi}_0$ are respectively analytic inside and outside an annular region Γ of the complex λ plane; and an arbitrary function of λ, $g_0(\lambda)$, satisfying $dg_0(\lambda) = 0$ in the annulus. One then defines

$$g(\lambda,x) = \widetilde{\Psi}_0^{-1}\, g_0(\lambda)\, \Psi_0 = \widetilde{\Psi}^{-1}\Psi \tag{9}$$

$$\text{i.e.} \quad g_0 = \widetilde{\Psi}_0 \widetilde{\Psi}^{-1} \Psi \Psi_0^{-1} .$$

Now, since $(\Psi_0, \widetilde{\Psi}_0, C_0)$ satisfies (8), from (9) one may write $\widetilde{\Psi}(dg(\lambda,x))\Psi^{-1}$ in two equivalent forms: the right and left-hand side of

$$\widetilde{\Psi} d \widetilde{\Psi}^{-1} - \widetilde{\Psi}\widetilde{\Psi}_0^{-1} C_0 \widetilde{\Psi}_0 \widetilde{\Psi}^{-1} = \Psi d \Psi^{-1} - \Psi \Psi_0^{-1} C_0 \Psi_0 \Psi^{-1} \tag{10}$$

Writing $\widetilde{\Psi} = \widetilde{\Psi}'\widetilde{\Psi}_0$, $\Psi = \Psi'\Psi_0$, (10) may be written:

$$\widetilde{\Psi}' d\, \widetilde{\Psi}'^{-1} = \Psi' d\, \Psi'^{-1} . \tag{11}$$

The right-hand side is analytic inside Γ, whereas the term on the left is analytic in the rest of the complex λ plane. We therefore have (by Liouville's theorem), a λ-independent matrix

$$A(x) = \widetilde{\Psi}' d\, \widetilde{\Psi}'^{-1} = - d\Psi'\Psi'^{-1} ,$$

satisfying $\quad (d + A)\Psi' = 0 ,$

$$\widetilde{\Psi}'^{-1}(d - A) = 0 .$$

So, the splitting of $g_0(\lambda)$ in (9) has yielded a new solution $(\Psi', \widetilde{\Psi}', A)$ to eq.(8) from the given one $(\Psi_0, \widetilde{\Psi}_0, C_0)$. The new solution, however, is not completely independent of the old one. For instance, the singularities of A coincide with those of C_0. Some technical details of this method may be found in [27].

The role of a linear system like (5) has also been emphasized by Zakharov et al [26,27,44], in their attempts to classify all models solvable by the Riemann-Hilbert method. Considering the system

$$\left(\partial_x + U(x,y,\lambda)\right)\Psi = 0 , \quad \left(\partial_y + V(x,y,\lambda)\right)\Psi = 0 , \tag{12}$$

with U,V being, for example, rational functions of a complex parameter λ :

$$U(\lambda;x,y) = \sum_{k=0}^{N}\sum_{i=0}^{\alpha_k} (\lambda-\lambda_k)^{-i}\, U_{ik}(x,y)$$
$$V(\lambda;x,y) = \sum_{k=0}^{M}\sum_{i=0}^{\beta_k} (\lambda-\mu_k)^{-i}\, V_{ik}(x,y) \quad ,$$

and observing that the form of (12) is left invariant by transformations analogous to gauge transformations:

$$V' = g^{-1}Vg + g^{-1}\partial_y g \quad , \quad U' = g^{-1}Ug + g^{-1}\partial_x g \, , \qquad (13)$$
$$\Psi' = g^{-1}\Psi \quad ,$$

they conjectured that all integrable models fall into ('gauge')equivalence classes. The transformations (13) form a group, the 'gauge' group, enabling one to classify all possible linear systems and to confine consideration to only one representative from each class, which may be chosen, using the gauge freedom, to be of the most convenient form.

These authors have also considered the possibility of generalizing the linear system (12) to obtain integrable systems in higher dimensions. They have suggested two fundamentally different ways of generalizing (12). The first involves the formal change $\lambda \rightarrow i\frac{\partial}{\partial t}$ in (12), where t is the third variable. This is clearly most easily achieved if U,V are polynomials in λ. The system (12) is then replaced by the system of equations for the matrix-valued function $\Psi(x,y,t)$:

$$\left[\partial_x + U\left(x,y,t,i\tfrac{\partial}{\partial t}\right)\right]\Psi = 0 \quad , \quad \left[\partial_y + V\left(x,y,t,i\tfrac{\partial}{\partial t}\right)\right]\Psi = 0 \quad .$$

This scheme incorporates the equation of Kadomtsev-Petviashvili and also the "three-wave problem" of nonlinear optics [26] ; both three dimensional equations. However, somewhat more interesting for gauge theories is their second method of generalizing (12) to higher dimensions. This replaces (12) by the first order system:

$$D_1\Psi = \sum_{k=0}^{N} \lambda^k \left(\beta_k\partial_k + U_k\right)\Psi = 0$$
$$D_2\Psi = \sum_{k=0}^{M} \lambda^k \left(\alpha_k\tilde{\partial}_k + V_k\right)\Psi = 0 \; ,$$

where $\partial_k, \tilde{\partial}_k$ denote differentiation with respect to generally independent variables $x_k, \tilde{x}_k$, of which there are (N+M+2) here; α_k, β_k are scalar functions which may be constant; and U_k, V_k are matrix functions of the (N+M+2)variables. As we shall see, the self-duality equations of pure gauge theories fall into this scheme.

In the next chapter we consider the theory of the principal chiral field in two dimensions, which has been found to be a useful model for gauge theories. Not only does this model mimic quantum properties of gauge theories (such as asymptotic freedom), but the structure of the classical theory is very similar to the self-dual sector of gauge theories; and this has, in recent years, motivated much work on self-dual fields. Our discussion will mainly be concerned with the infinite set of symmetries of the chiral field, which, as we shall demonstrate, has much to do with the existence of a Lax representation. We then demonstrate (in chapter 3) the similarity of this hidden symmetry structure of chiral fields to that of self-dual gauge fields. Remarkably, our discussion of self-dual fields can be generalized to the case of extended supersymmetric gauge theories; and we explicitly obtain, in chapter 4, an infinite set of continuity equations for these theories.

Some years ago, Polyakov pointed out that the loop space (functional) equations of three dimensional gauge theories were similar to a three dimensional chiral model, and based on this similarity he suggested the existence of an infinite set of symmetries of the loop space equations. In chapter 3 we show that loop space fields do indeed have a symmetry structure very similar to that of two dimensional chiral fields. Our discussion is based on the remarkable similarity of the loop space fields to chiral fields over a three dimensional space-time with one Killing vector. We also consider the equations for such chiral fields; and find them to be integrable.

As we have already emphasized, the 'zero curvature' integrability condition is a statement of the path independence of a phase factor $\psi = P\, e^{\int A \cdot dx}$, where A is the 'flat connection'. Motivated by this correspondence, we study (in chapter 5) the (non-)integrability ofthe full four dimensional Yang-Mills equations by considering the path-ordered phase factor of gauge theories [12] on a fixed, straight-line path. We attempt to determine the conditions under which this phase factor can be written as a product of local (path-independent) objects at the end-points of the path. Apart from the well-known case [15] where the path is restricted to lie on a null plane in complexified euclidean space (in which case the phase factor is integrable if the curvature is (anti-) self-dual), we fail to identify any further integrable sectors of the pure gauge theory. However, for the most general case, we obtain a remarkable approximate representation of the path-dependent phase factor, in which the non-integrability of the gauge connection manifests itself in a single local matrix at the mid-point of the straight-line path. If the phase factors around a lattice plaquette are thus approximated, we show that the correct continuum action results. Thus our representation of the phase factor effectively yields a formulation of the lattice action equivalent to Wilson's, in which the four link variables are replaced by a collection of eight local, path-independent ones.

The discovery of the integrability of the self-duality equations [15,16], which resulted in their remakable solution [17,18], was stimulated by the realization of BPST [19] that interesting solutions of the second-order Yang-Mills equations could be obtained by solving a set of algebraic equations for the field strength; i.e. the self-duality equations. In chapter 6, motivated by the example of self-duality in four dimensions, we search for first-order nonlinear equations for the potential which imply the second-order equations of higher dimensional gauge theories in the hope of finding integrable sectors of such theories. We show that in dimensions $4<d\leq 8$, an insistence upon the familiar sight of an algebraic equation for the components of the field strength yields interesting results. The most interesting are the sets of equations in eight dimensions which have a structure related to the algebra of the octonions.

Chapter 2: Hidden symmetry of the two dimensional chiral model.

(i) We consider the chiral model defined by the lagrangian

$$\mathcal{L} = \frac{1}{16} \operatorname{tr} \; \partial_\mu g^{-1}(x)\, \partial_\mu g(x) \tag{1}$$

(where $g(x)$ takes values in a compact lie group), which has, in terms of the pure-gauge lie algebra valued connection

$$A_\mu = g^{-1} \partial_\mu g \quad , \tag{2}$$

the equation of motion:

$$\partial_\mu A_\mu = 0 \quad . \tag{3}$$

For all models with equations of motion of the form (2,3), Brezin et al [20], following Luscher and Pohlmeyer [22,23], wrote down an algorithm for the construction of an infinite set of nonlocal conserved charges. Noting that any member of a hierarchy of conserved currents can be written in the form

$$J_\mu^{(n)} = \epsilon_{\mu\nu} \partial_\nu \chi^{(n)} \quad , \quad n \geq 1 \quad , \tag{4}$$

they noticed that such a hierarchy of currents can be generated iteratively by defining the $(n+1)$th current:

$$J_\mu^{(n+1)} = D_\mu \chi^{(n)} = (\partial_\mu + A_\mu)\chi^{(n)} \quad , \tag{5}$$

which is conserved if (3) is satisfied and if $\chi^{(n)}$ satisfies the equation:

$$D_\mu \partial_\mu \chi^{(n)} = 0 \quad . \tag{6}$$

From (4), we see that

$$D_\mu \partial_\mu \chi^{(n)} = D_\mu \epsilon_{\mu\nu} J_\nu^{(n)} = \epsilon_{\mu\nu} D_\mu D_\nu \chi^{(n-1)} \quad , \text{ using (5);}$$

which is clearly zero because A_μ is pure-gauge. Thus, provided (2,3) are satisfied, the sufficient condition for the conservation of $J_\mu^{(n+1)}$ is clearly that $J_\mu^{(n)}$ should be conserved. As Brezin et al noticed, this hierarchy of currents clearly exists, since the equation of motion (3) has the form of a continuity equation, allowing us to iteratively construct all the currents starting from $J_\mu^{(1)} = A_\mu$ and $\chi^{(0)} = 1$. This inductive proof of the conservation of an infinite hierarchy of currents, may be replaced [21] by considering a functional Q of the fields which is also a function of space-time satisfying

$$\lambda D_\mu Q = \epsilon_{\mu\nu} \partial_\nu Q \tag{7}$$

Then $j_\mu = \epsilon_{\mu\nu} \partial_\nu Q$ is clearly conserved; and generates infinitely many currents if Q has a power series expansion: $Q = \sum_{n=0}^{\infty} \chi^{(n)} \lambda^{(n)}$, where the coefficients $\chi^{(n)}$ satisfy the recurrence relation implicit in (4) and (5). We note that (7) is in fact a pair of linear equations

$$(\partial_1 - \lambda D_0)Q = 0 \tag{8}$$

$$(\partial_0 + \lambda D_1)Q = 0 \quad , \tag{9}$$

which are consistent if

$$[\partial_1 - \lambda D_0 , \partial_0 + \lambda D_1] = \lambda \partial_\mu A_\mu - \lambda^2 F_{01} = 0 \tag{10}$$

for all λ. In other words, we have a Lax representation for the system of equations (2,3), which is the key to Brezin et al's algorithm for the infinite series of conserved currents. This explicitly demonstrates the link between the linear system and an infinite set of conserved currents implied by Polyakov's contour argument. Writing (8,9) in the form

$$(\delta_{\mu\nu}\lambda - \epsilon_{\mu\nu})\partial_\nu Q = -\lambda A_\mu Q \quad , \tag{11}$$

and multiplying both sides by

$$\delta_{\nu\rho}(\delta_{\mu\nu}\lambda - \epsilon_{\mu\nu})^{-1} = \frac{1}{\lambda^2-1}(\lambda\delta_{\rho\mu} + \epsilon_{\rho\mu})$$

we obtain

$$\left(\partial_\rho + \frac{\lambda}{\lambda^2-1}(\lambda\delta_{\rho\mu} + \epsilon_{\rho\mu})A_\mu\right)Q = 0 \quad , \tag{12}$$

which is the form of the linear system first obtained by Pohlmeyer [24] (in light-cone coordinates) and used by Zakharov and Mikhailov [27] in their development of the Riemann-Hilbert method for this model. We note that (12) is a statement of Polyakov's dual transformation mapping A_μ to another pure-gauge $A_\mu(\lambda)$, a linear combination of A_μ and its dual $\epsilon_{\mu\nu}A_\nu$, in such a way that insisting on the zero curvature of $A_\mu(\lambda)$ is equivalent to the the equation of motion; (a situation reminiscent of four-dimensional self-duality). We may formally solve the $\rho = 1$ component of (12) by writing

$$Q(\lambda, x_0, x_1) = P \exp \int_{-\infty}^{x} dy \frac{\lambda}{\lambda^2-1}(\lambda A_1 - A_0) \quad , \tag{13}$$

$$Q(\lambda, x_0, -\infty) = 1 \quad .$$

Now, $Q(\lambda, x_0, +\infty)$ is clearly time-independent if we assume the boundary conditions $\lim_{x_1 \to \pm\infty} A_\mu(x_0, x_1) = 0$; and expanding the exponential in a power series in λ yields

$$1 + \lambda \int_{-\infty}^{\infty} dx\, A_0(t,x) + \lambda^2 \int_{-\infty}^{\infty} dx' \left[A_1(t,x') + A_0(t,x') \int_{-\infty}^{x'} dx''\, A_0(t,x'') + O(\lambda^3)\right. \tag{14}$$

The coefficients of λ, λ^2 correspond to the first two nonlocal currents of [22] . In fact we may write

$$Q(\lambda, x_0, +\infty) = e^{q(\lambda)} \quad ; \quad q(\lambda) = \sum_{n=1}^{\infty} \lambda^n q^{(n)} ,$$

then $q_a^{(n)} \equiv \mathrm{tr}\ t_a q^{(n)}$ yields the charges of [22] , where t_a are the generators of the Lie algebra. Following [21] , we also note that from any given solution $A_\mu(x) = g^{-1}\partial_\mu g$, we may construct, from (11), a new one depending on λ :

$$A_\mu(\lambda; x) = Q(x)\left(\partial_\mu - \frac{1}{\lambda}\, \epsilon_{\mu\nu}\partial_\nu\right) Q^{-1}(x) \quad . \tag{11'}$$

It is clear, using (12), that $\partial_\mu A_\mu(\lambda; x) = 0$. This transformation is just the 'dual transformation' [29] which induces the symmetry responsible for the above conserved charges.

(ii) We have seen that the matrix $Q(\lambda, x_0, +\infty)$, known as the monodromy matrix [3], which connects solutions of the linear system at plus and minus (spatial) infinity, is entirely time independent; <u>all</u> its matrix elements are conserved charges [22] . This is not a very desirable feature, since it is uncharacteristic of the usual completely integrable models. In the usual case [4] , of the KdV or sine-Gordon equations for instance, the action-angle variables are obtained directly from the monodromy matrix of the associated linear system. Action-angle variables are not known for the chiral field theories. However, the monodromy matrix, as we have seen, serves as a generating functional of an infinite number of nonlocal conserved charges, which do not commute amongst themselves. Now, just as the infinite-dimensional abelian symmetry algebra (generated by the angle variables) is related to integrability in the usual case, these nonlocal charges raise the possibility that complete integrability of such field theories can be related to the existence of an infinite dimensional non-abelian symmetry algebra. It is known, however, that these nonlocal charges do <u>not</u> form a Lie algebra [30,31] . However, there does exist an infinite dimensional Lie algebra of symmetry transformations acting on the space of solutions of the field equation

$$\partial_\mu A_\mu = \partial_\mu\left(g^{-1}\partial_\mu g\right) = 0 \quad . \tag{15}$$

We suppose that the infinitesimal transformation:

$$g' = g + \delta g = g - g\,S(x) \qquad (16)$$

is a symmetry of (15); where the infinitesimal S(x) is a Lie algebra-valued matrix. For this to be the case, to first order, S must satisfy

$$\partial_\mu [D_\mu , S] = \partial^2 S + [A_\mu , S] = 0 \quad . \qquad (17)$$

We note that (17) is automatically satisfied if

$$[D_\mu , S] = \epsilon_{\mu\nu}\, \partial_\nu\, \eta(x) \qquad (18)$$

for any $\eta(x)$. This equation, and hence (17), imposes no extra restrictions on g(x) if S and η depend on a parameter λ in such a way that their power series expansions satisfy the relation

$$\eta = \frac{1}{\lambda} S$$

Then (18) is precisely the linear system for the Lie algebra valued function S having the field equations as compatibility conditions 41 ; and corresponding to eq.(11) for the group-valued function Q. Thus every solution of

$$\left[(\partial_\mu - \lambda\, \epsilon_{\mu\nu} D_\nu), S \right] = 0 \qquad (19)$$

yields a symmetry of the equations of motion.

We note that the transformation (16) generates a new solution to the field equations from an old one if S satisfies (19). From (16) we have that

$$g^{-1} g' = \mathbb{1} - S$$

and the change in A is given by

$$g'^{-1} \partial_\mu g' - g^{-1} \partial_\mu g = [D_\mu , S]$$

$$= \frac{1}{\lambda}\, \partial_\nu S\, \epsilon_{\mu\nu} \quad , \text{ from (19).}$$

Therefore,

$$g'^{-1} \partial_\mu g' - g^{-1} \partial_\mu g = \frac{1}{\lambda}\, \epsilon_{\mu\nu}\, \partial_\nu (g^{-1} g') \quad .$$

This is precisely the Backlund transformation of [39] linking two solutions g and g'. We remark that just as g and g' are related, two solutions Q, Q' of (11,11') are related by

$$(\partial_\mu - \lambda\, \epsilon_{\mu\nu}\, \partial_\nu)(Q^{-1} Q') = 0 \quad .$$

The function S, like Q , depends on the parameter λ ; and we may expand S in a power series:

$$S = \sum_{n=0}^{\infty} \Lambda^{(n)} \lambda^n \quad , \qquad (20)$$

yielding, through eqs.(16–19), infinitely many nonlocal symmetries of (15). We note that this 'hidden symmetry' has been derived, just as the nonlocal continuity equations were previously derived, using the linear system; and in what follows, we draw attention to the importance of the linear system for this hidden symmetry. In particular, we shall show that it is the symmetry of the linear system which is the source of this hidden symmetry of the field equations. In order to emphasize the role of the linear system, we initially consider functions Q and S satisfying just one of the equations in (11) and (19) respectively; namely, we impose

$$\tilde{\partial}(\lambda)\, Q \equiv (\partial_1 - \lambda\partial_0)\, Q = \lambda A_0 Q \tag{21}$$

and

$$(\partial_1 - \lambda\partial_0)\, S = \lambda\, [A_0, S] \quad . \tag{22}$$

Analogously to (13), (21) has the formal solution

$$Q(\lambda;x) = P \exp \int_{-\infty}^{\tilde{x}} d\tilde{x}(\lambda)\ \lambda A_0 \quad . \tag{23}$$

Further, we may clearly write

$$Q(\lambda,x) = \exp\ Z(\lambda,x) \quad ,$$

where the power series expansion of the Lie algebra valued function

$Z(\lambda,x) = \sum_{N=1}^{\infty} X_{(N)}\lambda^N$ yields

$$Q(\lambda;x) = \lim_{N\to\infty} Q_N = \lim_{N\to\infty} e^{X^{(N)}\lambda^N}\, e^{X^{(N-1)}\lambda^{N-1}} \cdots\cdots\cdots e^{X^{(2)}\lambda^2}\, e^{X^{(1)}\lambda} \,. \tag{24}$$

By insisting on the consistency of equations (21) and (22), we shall now find a remarkable expression for S in terms of Q ; and in the process, we shall relate all the $X^{(N)}$'s recursively to A_0, thus obtaining through (24), an explicit representation for the path-ordered object in (23). Inserting the expansion (20) into (22), we obtain the recurrence relation [32]

$$\partial_1 \Lambda^{(n)} = \partial_0 \Lambda^{(n-1)} + [A_0, \Lambda^{(n-1)}] \tag{25}$$

Now, assuming the transformation corresponding to $\Lambda^{(0)}$ to be just a global gauge transformation, i.e. $\Lambda^{(0)} = T$, a constant element of the Lie algebra, we obtain from (25) for n = 1 :

$$\partial_1 \Lambda^{(1)} = [A_0, T]$$

We also note, from (21) & (24), that

$$\partial_1 X^{(1)} = A_0 \quad , \tag{26}$$

so $$\Lambda^{(1)} = [X^{(1)}, T] \equiv [X, T] \; . \tag{27}$$

(In what follows we denote $X^{(1)}$ by X). For $n = 2$, we find :

$$\begin{aligned}\partial_1 \Lambda^{(2)} &= \partial_0 \Lambda^{(1)} + [\partial_1 X, \Lambda^{(1)}] \\ &= [\partial_0 X, T] + [\partial_1 X, [X, T]] \quad \text{, from (26);} \\ &= [\partial_0 X, T] + \tfrac{1}{2}[[\partial_1 X, X], T] + \tfrac{1}{2}\partial_1 [X, [X, T]] ,\end{aligned}$$

which we may write as

$$\partial_1 \Lambda^{(2)} = \tfrac{1}{2}\partial_1 [X,[X,T]] + \partial_1 [X^{(2)}, T] \; ,$$

yielding

$$\Lambda^{(2)} = \tfrac{1}{2} [X,[X,T]] + [X^{(2)}, T] \tag{28}$$

if $X^{(2)}$ is defined by

$$\partial_1 X^{(2)} = \partial_0 X + \tfrac{1}{2}[\partial_1 X, X] \; . \tag{29}$$

Equations (26-29) reproduce the results of [32,33] . However, these expressions do not provide sufficiently many terms of the series in (20) to precisely determine the structure of S. We note, however, that (29) is indeed the form of $X^{(2)}$ given by (21,24), confirming the validity of the representation for $Q(\lambda, x)$ given by (24). Using (26), we rewrite (21) as

$$Q(\partial_1 - \lambda \partial_0) Q^{-1} = -\lambda \partial_1 X \tag{30}$$

where

$$Q^{-1}(\lambda; x) = \lim_{N\to\infty} Q_N^{-1} = \lim_{N\to\infty} e^{-X\lambda} e^{-X^{(2)}\lambda^2} \cdots\cdots\cdots e^{-X^{(N)}\lambda^N} \; . \tag{31}$$

This equation, as demonstrated by (29), clearly provides an expression for $X^{(N)}$ in terms of all the $X^{(m)}$, $m<N$; and thus ultimately in terms of A_0(by (26)). Now, expanding the l.h.s. of (30) up to terms of $O(\lambda^4)$, we find

$$\begin{aligned}Q_4(\partial_1 - \lambda\partial_0) Q_4^{-1} = & -\lambda \partial_1 X + \lambda^2\left(-\tfrac{1}{2}[X, \partial_1 X] - \partial_1 X^{(2)} + \partial_0 X\right) \\ & + \lambda^3\Big(-\tfrac{1}{6}[X[X, \partial_1 X]] - [X^{(2)}, \partial_1 X] - \partial_1 X^{(3)} + \tfrac{1}{2}[X, \partial_0 X] \\ & \qquad + \partial_0 X^{(2)}\Big) \\ & + \lambda^4\Big(+\tfrac{1}{4!}[X[X[\partial_1 X, X]]] - \tfrac{1}{2}[X^{(2)}, \partial_1 X^{(2)}] - \tfrac{1}{2}[X^{(2)}, [X, \partial_1 X]] \\ & \qquad - [X^{(3)}, \partial_1 X] - \partial_1 X^{(4)} + \tfrac{1}{6}[X[X, \partial_0 X]] \; .\end{aligned} \tag{32}$$

Now, comparing with (30) we see that only the $O(\lambda)$ term on the right is necessary; so for (30) to be true for all λ, we need to set the coefficients of all the higher powers of λ to zero individually. This yields (29) and

$$\partial_1 X^{(3)} = -\frac{1}{6}[X[X,\partial_1 X]] - [X^{(2)},\partial_1 X] + \frac{1}{2}[X,\partial_0 X] + \partial_0 X^{(2)}, \qquad (33)$$

$$\partial_1 X^{(4)} = -\frac{1}{4!}[X[X[X,\partial_1 X]]] - \frac{1}{2}[X^{(2)},\partial_1 X^{(2)}] - \frac{1}{2}[X^{(2)},[X,\partial_1 X]] - [X^{(3)},\partial_1 X] + \frac{1}{6}[X[X,\partial_0 X]] . \qquad (34)$$

We proceed to determine $\Lambda^{(3)}$:

$$\begin{aligned}\partial_1 \Lambda^{(3)} &= \partial_0 \Lambda^{(2)} + [\partial_1 X, \Lambda^{(2)}] \\ &= \frac{1}{2}[\partial_0 X,[X,T]] + \frac{1}{2}[X,[\partial_0 X,T]] + [\partial_0 X^2, T] \\ &\quad + \frac{1}{2}[\partial_1 X,[X,[X,T]]] + [\partial_1 X,[X^{(2)},T]] \quad , \text{ using (28);} \\ &= \frac{1}{6}\partial_1[X,[X,[X,T]]] + \partial_1[X,[X^{(2)},T]] + [\partial_0 X^{(2)},T] \\ &\quad - [[X^{(2)},\partial_1 X],T] + \frac{1}{2}[[X,\partial_0 X],T] - \frac{1}{6}[[X,[X,\partial_1 X]],T]\end{aligned}$$

using (29) and the Jacobi identity. Now, since $\partial_1 X^{(3)}$ is given by (33) we clearly have

$$\Lambda^{(3)} = \frac{1}{6}[X,[X,[X,T]]] + [X^{(2)},[X,T]] + [X^{(3)},T] \qquad (35)$$

and we similarly find

$$\Lambda^{(4)} = \frac{1}{4!}[X,[X,[X,[X,T]]]] + \frac{1}{2}[X^{(2)},[X,[X,T]]] + \frac{1}{2}[X^{(2)},[X^{(2)},T]] + [X^{(3)},[X,T]] + [X^{(4)},T] . \qquad (36)$$

We are now in a position to write the generating function (20) of the $\Lambda^{(n)}$'s in terms of a x_μ and λ-dependent similarity transformation of the constant matrix T [35]:

$$S(\lambda, x) = Q(\lambda, x)\, T\, Q^{-1}(\lambda, x) \quad , \qquad (37)$$

a form which clearly makes (22) consistent with (21); and which may explicitly be checked with (27,28,35,36) by using the Campbell-Hausdorff formula:

$$\begin{aligned}e^{-\lambda A} B e^{\lambda A} &= B - \lambda[A,B] + \frac{\lambda^2}{2!}[A,[A,B]] - \frac{\lambda^3}{6}[A,[A,[A,B]]] + \cdots \\ &= B + \lambda[B,A] + \frac{\lambda^2}{2!}[[B,A],A] + \frac{\lambda^3}{6}[[[B,A],A],A] + \cdots\end{aligned}$$

We now use the form (37) of the function S to define the field transformations (16). We first consider the change in the lagrangian density due to the infinitesimal transformation (16) where S is defined by (37) and Q satisfies just (21).

$$\begin{aligned}
\delta\mathcal{L} &= \tfrac{1}{16}\,\mathrm{tr}\left[\partial_\mu(g^{-1}+Sg^{-1})\partial_\mu(g-gS)-\partial_\mu g^{-1}\partial_\mu g\right] \\
&= \tfrac{1}{8}\,\mathrm{tr}\,A_\mu\partial_\mu S \\
&= \tfrac{1}{8}\,\mathrm{tr}\left(A_0\partial_0 S+\lambda A_1\partial_0 S+\lambda[A_1,A_0]S\right) \quad \text{, using (22) \& the cyclic property of the trace;} \\
&= \tfrac{1}{8}\,\mathrm{tr}\left\{A_0(\partial_0 S+\lambda\partial_1 S)+\lambda\left[\partial_0(A_1 S)-\partial_1(A_0 S)\right]\right\} \quad \text{, using } F_{01}\equiv 0\,; \\
&= \tfrac{1}{8}\,\mathrm{tr}\left\{-\tfrac{1}{\lambda}\,Q(\partial_1-\lambda\partial_0)Q^{-1}(\partial_0+\lambda\partial_1)S+\lambda\left[\partial_0(A_1S)-\partial_1(A_0S)\right]\right\} \quad \text{, using (21);} \\
&= \tfrac{1}{8}\,\mathrm{tr}\left\{-\tfrac{1+\lambda^2}{\lambda}\left[\partial_1 Q^{-1}\partial_0 Q-\partial_0 Q^{-1}\partial_1 Q\right]T+\epsilon_{\mu\nu}\,\lambda\,\partial_\mu(A_\nu S)\right\} \\
&= \tfrac{1}{8}\,\partial_\mu\,\epsilon_{\mu\nu}\,\mathrm{tr}\left\{\lambda A_\nu S+\left(\lambda+\tfrac{1}{\lambda}\right)Q^{-1}\partial_\nu Q\,T\right\}\,,
\end{aligned} \tag{38}$$

a total divergence.

The usual argument [33, 42], has been that the infinitesimal transformation is a symmetry of the action since one may ignore the surface terms which result from the integration of (38) over space-time, by choosing appropriate boundary conditions. Indeed, (38) vanishes at spatial infinity if $\lim\limits_{x\to\pm\infty} A_\mu = 0$, since then $\lim\limits_{x\to\pm\infty} Q = 0$ also. However, as has been recently pointed out [31], it is these usually ignored surface terms which vitiate this argument; since one may not impose consistent boundary conditions on A_μ and Q which are themselves invariant under the transformation. We note that if the equations of motion are imposed, then the action is indeed invariant since then

$$\delta\mathcal{L} = \tfrac{1}{8}\,\mathrm{tr}\;\partial_\mu(A_\mu S)\,,$$

and the usual boundary conditions $\lim\limits_{x\to\pm\infty} A_\mu = 0$ suffice. The resulting (on-shell) conserved charges are those of [22]. Thus in the general case of a non-compact space-time, the original belief [22], that these conservation laws are dynamic rather than algebraic statements in the sense that they are not associated with a Noether symmetry of the original Lagrangian, but come directly from the solution space of the system, still holds good. That care needs to be taken with surface terms in the Noether construction when dealing with dynamical conservation laws has also been emphasized by Chodos [40].

We have seen that the conservation laws of [22] are related to a set of symmetry transformations on the solution space of the model. We now show that these symmetry transformations close under an infinite dimensional Lie algebra. We write the infinitesimal transformation (16) in the form

$$\delta_a g = -g S_a \tag{39}$$

where the $S_a = Q T_a Q^{-1}$, $a = 1,\ldots\ldots,N$, form a basis of the Lie algebra G. We define, following [34], the infinitesimal operators of the symmetry group:

$$M_a(\lambda) = -\int d^2y \; \delta_a g \frac{\delta}{\delta g} = \int d^2y \; g \, S_a(\lambda) \frac{\delta}{\delta g(y)} , \tag{40}$$

which clearly have the structure of Killing fields in a local functional basis. The composition of two such symmetry operations is clearly given by the Lie bracket:

$$\begin{aligned} [M_a(t), M_b(r)] &= \mathcal{L}_{M_a(t)} M_b(r) \\ &= \int d^2y \left[M_a(t), g(y) S_b(r) \frac{\delta}{\delta g(y)} \right] . \end{aligned} \tag{41}$$

The commutator on the right clearly contains two pieces: one involving the functional variation of g :

$$\left[\frac{\delta}{\delta g(x)}, g(y) \right] = \delta(x-y)$$

and the other due to the change induced in $S_a \equiv S_a[g]$, a functional of $g(x)$, by the transformation (39). We may therefore write

$$\begin{aligned} [M_a(t), M_b(r)] &= \int d^2y \; g(y) [S_a(t), S_b(r)] \frac{\delta}{\delta g(y)} \\ &\quad - \int d^2y \; g(y) [\delta_a S_b(r) - \delta_b S_a(t)] \frac{\delta}{\delta g(y)} \\ &= -\int d^2y \; g(y) \{ \delta_a S_b(r) - \delta_b S_a(t) - [S_a(t), S_b(r)] \} \frac{\delta}{\delta g(y)} , \end{aligned} \tag{42}$$

which is clearly equal to

$$-\int d^2y \; [\delta_a(t), \delta_b(r)] \, g(y) \frac{\delta}{\delta g(y)} \qquad , \text{ from (39).}$$

Here, the change induced in $S_b(r)$ by the infinitesimal transformation $\delta_a g = -g S_a(t)$ is given by

$$\delta_a S_b(r) \equiv \delta_a(t) S_b[r; g] = S_b[r; g - g S_a(t)] - S_b[r; g] . \tag{43}$$

We also note that

$$\begin{aligned} \delta_a S_b &= \delta_a (Q T_b Q^{-1}) = \delta_a Q \, T_b Q^{-1} + Q T_b \delta_a Q \\ &= [\delta_a Q . Q^{-1}, S_b] . \end{aligned} \tag{44}$$

Now $\quad \delta_a(t) \chi^{(1)} = \chi^{(1)}(g + \delta_a g) - \chi^{(1)}(g)$

is given by:

$$\int^{X} dy \left(A_0 (g + \delta_a g) - A_0(g) \right) = - \int^{X} D_0 S_a(t) \, dy \ .$$

Therefore, using (22), we obtain

$$\delta_a(t) \, X^{(1)} = - \frac{S_a(t)}{t} \ . \tag{45}$$

We may therefore write $\delta_a Q$ in terms of S , since all the coefficients in the power series expansion of $Q(\lambda)$ depend recursively on $X^{(1)}$. Using eqs. (29-36,45), we find that

$$\begin{aligned}
\delta_a(t) \, \Lambda^{(1)} &= - \frac{1}{t} \left[S_a(t), T \right] \\
\delta_a(t) \, \Lambda^{(2)} &= - \frac{1}{t^2} \left[S_a(t), T \right] - \frac{1}{t} \left[S_a(t), \left[X^{(1)}, T \right] \right] \\
\delta_a(t) \, \Lambda^{(3)} &= - \frac{1}{t^3} \left[S_a(t), T \right] - \frac{1}{t^2} \left[S_a(t), \left[X^{(1)}, T \right] \right] - \frac{1}{t} \left[S_a(t), \left[X^{(2)}, T \right] \right] \\
&\quad - \frac{1}{2t} \left[S_a(t), \left[X^{(1)}, \left[X^{(1)}, T \right] \right] \right] \ .
\end{aligned} \tag{46}$$

Therefore,

$$\delta_a(t) \, S(r) = \sum_{n=0}^{\infty} \delta_a(t) \, \Lambda^{(n)} r^n$$

to third order in r is just :

$$- \left(\frac{r}{t} + \left(\frac{r}{t}\right)^2 + \left(\frac{r}{t}\right)^3 \right) \left(\left[S_a(t), T \right] + r \left[S_a(t), \left[X^{(1)}, T \right] \right] + r^2 \left[S_a(t), \left[X^{(2)}, T \right] \right] + r^2 \frac{1}{2} \left[S_a(t), \left[X^{(1)}, \left[X^{(1)}, T \right] \right] \right] \right) \ ; \quad r \neq t \ .$$

Now, since

$$\sum_{n=1}^{\infty} \left(\frac{r}{t}\right)^n = \left(1 - \frac{r}{t}\right)^{-1} - 1 = \frac{t}{t-r} - 1 = \frac{r}{t-r} \ ,$$

we may make the identification :

$$\delta_a(t) \, S_b(r) = - \frac{r}{t-r} \left[\left[S_a(t), S_b(r) \right] - c_{abc} \, S_c(r) \right] \ , \text{for all } t, r \ ; \tag{47}$$

where C_{abc} are the structure constants of the Lie algebra G. We may write (47) in the form :

$$\delta_a(t) \, S_b(r) = - \frac{r}{t-r} \left[\left(S_a(t) - S_a(r) \right), S_b(r) \right] \ , \tag{48}$$

which allows us, using (44), to identify the change in Q :

$$\begin{aligned}
\delta_a(t) \, Q(r) &\equiv Q\left[r ; g - g \, S_a(t) \right] - Q\left[r ; g \right] \\
&= - r \left(\frac{S_a(t) - S_a(r)}{t-r} \right) Q(r) \ .
\end{aligned} \tag{49}$$

We are now in a position to observe that this hidden symmetry is a symmetry of the linear system. We note that up to first order in the variation $\delta_a g = -g S_a(t)$, the l.h.s. of eq. (21) is

$$\left(\partial_1 - r\partial_0 - r\left(A_0 + \delta_a(t) A_0\right)\right)\left(Q(r) + \delta_a(t) Q(r)\right)$$

$$= \left(\partial_1 - r\partial_0 - rA_0\right)\delta_a(t) Q(r) - r\delta_a(t) A_0 Q(r) \qquad \text{, if Q satisfies (21);}$$

$$= \left(\partial_1 - r\partial_0 - rA_0\right)\delta_a(t) Q(r) - \frac{r}{t}\partial_1 S_a(t)\, Q(r) \quad ,$$

(since $\delta_a(t) A_0 = -D_0 S_a(t) = -\frac{1}{t}\partial_1 S_a(t)$),

which vanishes if $\delta_a Q$ has the form given by (49), confirming that the linear equation (21) is invariant under the transformation (16). Similarly, one may also show that $\delta_a S_b$ given by (48) satisfies the equation

$$\partial_1 \delta_a S_b = r\partial_0 \delta_a S_b + r\delta_a\left([A_0, S_b]\right) \quad ,$$

obtained by varying equation (22). This proves that the infinitesimal hidden symmetry of the equations of motion is due to an infinitesimal symmetry of the linear system.

We now return to (42), which on insertion of (47) gives

$$[M_a(t), M_b(r)] = c_{abc}\int d^2y \; g\left[\frac{tS_c(t) - rS_c(r)}{t-r}\right]\frac{\delta}{\delta g(y)} \tag{51}$$

Now, if we write

$$M_a(\lambda) = \sum_{n=0}^{\infty} M_a^n \lambda^n \tag{52}$$

a comparison of the coefficients of $t^m r^n$ on both sides of (51) immediately yields the commutation relations:

$$[M_a^m, M_b^n] = c_{abc} M_c^{n+m} \quad , \quad n,m \geqslant 0 \; . \tag{53}$$

The matrices

$$M_a^m = \frac{1}{m!}\frac{d^m}{d\lambda^m} M_a(\lambda)\Big|_{\lambda=0}$$

therefore realize a representation of the loop algebra $G \otimes \mathbb{R}[\lambda]$, where $\mathbb{R}[\lambda]$ is the algebra of the formal power series in λ. This derivation of the loop algebra clarifies and simplifies that originally presented by Dolan [34]; and since we have emphasized the role of the linear system, our discussion may easily be generalized to other models with similar linear systems. This is demonstrated by the work of Eichenherr [43], who has discussed the hidden symmetry algebra of the Heisenberg model,

$$\partial_t S = \frac{1}{2i}\partial_x[S, \partial_x S] \quad ; \quad S^2 = \mathbb{1} \ , \ S \in su(2) \ ; \tag{54}$$

(a model which is similar to the chiral model in the sense that it has a global group-invariance which may be generalized to an infinite parameter nonlocal invariance). Eichenherr has shown that the hidden symmetry algebra of (54) is directly carried over to the non-linear Schrodinger equation :

$$i\partial_t u + \partial_x^2 u + 2|u|^2 u = 0 \qquad , \ (u(x,t) \text{ a complex field}). \tag{55}$$

Now, (54) is 'gauge equivalent' to (55) [44] in the sense that the flat connections in the two linear systems may be mapped to each other by a 'gauge' transformation. That the hidden symmetry structure of (55) can be seen to have its origins in that of (54) further emphasizes that it is the symmetry of the linear system which is responsible for the symmetry of the equations of motion. Further, it suggests that this feature is common to all the models which are classified under the scheme of [26,27] .

The loop algebra $G \otimes \mathbb{R}[\lambda]$ has also been identified by Ueno and Nakamura [37,38] in the context of the Riemann-Hilbert problem which yields as a by-product, a formulation of the symmetries in terms of contour integrals. Using these, the verification of the commutation relations (53) is particularly simple [38] . We also note that much of the structure that we have displayed in this chapter has been duplicated in the literature [36]. In an interesting further development, Wu [45] has extended the symmetry algebra to $G \otimes \mathbb{R}[\lambda, \lambda^{-1}]$; a factor algebra of the Kac-Moody algebra over a one-dimensional centre. Wu's approach, translated to our notation, is as follows.

Above we have considered S to be a series in positive powers of λ :

$$S(x) = \sum_{n=0}^{\infty} \lambda^n \Lambda^{(n)} = \sum_{n=0}^{\infty} \lambda^{-n} \Lambda^{(-n-1)} \qquad .$$

We may also have additional symmetry transformations

$$\hat{\delta} g = -g \ R(x,\lambda)$$

with
$$R(x,\lambda) = S(x, \tfrac{1}{\lambda}) = \sum_{n=0}^{\infty} \lambda^n \Lambda^{(-n-1)}$$
$$= W(\lambda) \, T \, W(\lambda)^{-1} \qquad ,$$

where

$$W = \exp \sum_{n=0}^{\infty} \lambda^n \chi^{(-n-1)} \quad \text{satisfies}$$

$$(\partial_1 - \tfrac{1}{\lambda} D_0) W = 0$$

and

$$(\partial_0 + \tfrac{1}{\lambda} D_1) W = 0 \quad .$$

Now, just as we obtained equation (49),

$$\delta_a Q_b(r) = \frac{r}{t-r} \left(S_a(t) - S_a(r) \right) Q_b(r) \ ,$$

we may obtain, following [45],

$$\tilde{\delta}_a W_b(r) = \frac{r}{t-r} \left(R_a(t) - R_a(r) \right) W_b(r)$$

$$\delta_a W_b(r) = \frac{1}{1-tr} \left(S_a(t) - R_a(r) \right) W_b(r)$$

$$\tilde{\delta}_a Q_b(r) = \frac{tr}{1-tr} \left(R_a(t) - S_a(r) \right) Q_b(r) \ .$$

Using these, one may define the components of a Laurent expandable generator of infinitesimal transformations

$$M_a(\lambda) = \sum_{-\infty}^{+\infty} \lambda^m M^{(m)}$$

in the following fashion :

$$M_a^{(0)} = - \int d^2y \ \left(\delta_a^{(0)} + \tilde{\delta}_a^{(0)} \right) g \ \frac{\delta}{\delta g}$$

$$M_a^{(n)} = - \int d^2y \ \left(\delta_a^{(n)} g \right) \frac{\delta}{\delta g}$$

$$M_a^{(-n)} = - \int d^2y \ \left(\tilde{\delta}_a^{(n)} g \right) \frac{\delta}{\delta g} \qquad , \ n \geqslant 1 \quad .$$

These generators satisfy the commutation relations:

$$\left[M_a^{(n)} , M_b^{(m)} \right] = C_{abc} \ M_c^{(n+m)} \qquad ; \quad n, m \in \mathbb{Z} \ .$$

To conclude this discussion, we note that although the above infinitesimal field transformations form a Lie algebra, they are surprisingly not canonical transformations [31] . Conversely, the transformations generated by the non-local charges are obviously (by definition) canonical; but these do not form a Lie algebra [31,30] , contrary to the conjecture of [106]. This situation is a consequence of the same problem which prevents the construction of action-angle variables for this model in the routine fashion. Faddeev [3] has noted that this difficulty (of constructing action-angle variables) stems from the 'non-ultra-locality' of the linear system; i.e. the linear system contains derivatives of the canonical variables. This results in the impossibility of consistently defining the Poisson bracket of two monodromy matrices [30], which in turn implies that the algebra of the non-local charges is not a Lie algebra.

Although action-angle variables have not been found for the chiral models, it is certain that these are hamiltonian integrable systems since they display the lack of complete randomization which characterizes such systems (‡). For the O(N) sigma-model, Luscher and Pohlmeyer [22] have explicitly demonstrated that classical spinwaves do not decay into a superposition of abelian waves for large times, and that in fact, a generic solution decomposes into a set of massless lumps. They considered energy-momentum conservation in Minkowski-space light-cone coordinates:

$$\partial_+ T_- = 0 \quad , \quad \partial_- T_+ = 0$$

where $T_- = \frac{1}{2}\left(T_{00} + T_{01}\right) \quad , \quad T_+ = \frac{1}{2}\left(T_{00} - T_{01}\right)$

are the only two independent components of the energy-momentum tensor :

$$T_{\mu\nu} = \partial_\mu q^a \partial_\nu q^a - \frac{1}{2} g_{\mu\nu} \partial_\lambda q^a \partial^\lambda q^a \; ; \quad T^\mu_{\ \mu} = 0 \, , \quad T_{\mu\nu} = T_{\nu\mu} \, .$$

They argued that energy-momentum conservation means that, for instance, T_+ depends on x_+ only, so that energy flowing from right to left runs with exactly the speed of light without dispersion. This absence of dispersion definitely points to the existence of implicit integrals of motion, and possibly to integrability as well, since in general, dispersion of any finite amount of energy is required by statistical mechanics; a general initial condition should display a tendency towards the equipartition of energy with respect to the degrees of freedom (i.e. a tendency towards stochastization). Further, just as the non-integrability of a system is indicated by the inelasticity of collisions [46], the results of the

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(‡) Recently, moreover, Dikii [107] has made some progress on the construction of the integrals of motion for the chiral field equations.

Zamolodchikovs [28] on collisions of solitons seem to be sufficienly strong conditions for the complete integrability of the chiral field. Indeed, they used the nonlocal charges of the theory to obtain the exact S-matrix of the O(3) model. This obeys the factorization equations of Yang and Baxter [47] , demonstrating that the nonlocal charges play the role of action-angle variables in that they constrain and even determine the scattering of particles.

To conclude, we note that whereas the classical nonlocal charges do not form a Lie algebra, the corresponding quantum charges do [30] . This suggests that the quantum theory might actually be more tractable than the classical one considered here.

(iii) We append this discussion of hidden symmetries with the observation that energy-momentum conservation also yields an infinite set of (on-shell) non-local conserved currents.

We note that

$$T_{\mu\nu} = \mathrm{tr}\left\{\partial_\mu g\, \partial_\nu g^{-1} - \tfrac{1}{2}\, g_{\mu\nu}\, \partial_\rho g\, \partial_\rho g^{-1}\right\}$$

under a first-order variation :

$$\begin{aligned} T'_{\mu\nu} &= \mathrm{tr}\left\{\partial_\mu g' \partial_\nu g^{-1} + \partial_\mu g\, \partial_\nu g'^{-1} - \tfrac{1}{2}\, g_{\mu\nu}\left(\partial_\rho g' \partial_\rho g^{-1} + \partial_\rho g\, \partial_\rho g'^{-1}\right)\right\} \\ &= T_{\mu\nu} + \delta T_{\mu\nu} \end{aligned}$$

is also conserved provided both g and g' are solutions.

Writing, as before, $g' = g + \delta g = g - gS$, we find

$$\delta T_{\mu\nu} = \mathrm{tr}\left\{ \partial_\mu g\, \partial_\nu (S g^{-1}) - \partial_\mu (g S)\, \partial_\nu g^{-1} - g_{\mu\nu}\, A_\rho \partial_\rho S \right\} .$$

Then

$$\partial_\mu(\delta T_{\mu\nu}) = \mathrm{tr}\left\{ A_\mu \partial_\mu \partial_\nu S + \partial_\mu A_\nu \partial_\mu S + A_\nu \partial^2 S - \partial_\nu (A_\rho \partial_\rho S)\right\}$$

Now the third term

$$\begin{aligned} \mathrm{tr}\ A_\nu \partial^2 S &= -\mathrm{tr}\ A_\nu [A_\mu, \partial_\mu S] \quad \text{(using (17))} \\ &= \mathrm{tr}\ [A_\mu, A_\nu]\, \partial_\mu S \ , \end{aligned}$$

and using the identity $F_{\mu\nu} = 0$, it is clear that $\partial_\mu(\delta T_{\mu\nu}) = 0$.

The implications of these continuity equations are not clear, apart from the implication that the non-local symmetry transformations we have discussed commute with conformal transformations. We remark that these new conservation laws bear a striking resemblance to similar infinite sets of conservation laws in free field theories and electrodynamics (see e.g. [49]).

Chapter 3: Hidden symmetry of the integrable sectors of pure gauge theories.

The hidden symmetry structure of the chiral model discussed in the previous chapter is essentially carried over to both the self-duality equations and Polyakov's loop-space (functional) equations for three dimensional gauge theories.

3.1 : Self-dual gauge fields.

The self-duality equations $\quad F_{\mu\nu} = {}^{*}F_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$ (1)

in complexified euclidean space with coordinates

$$x = x_4 + i\underline{x}\cdot\underline{\sigma} = \begin{pmatrix} x_4 + ix_3 & x_2 + ix_1 \\ -x_2 + ix_1 & x_4 - ix_3 \end{pmatrix} \equiv \begin{pmatrix} y & -\bar{z} \\ z & \bar{y} \end{pmatrix} ,$$

(where the bar denotes complex conjugation if x is real), take the form [50]

$$F_{yz} = 0 = F_{\bar{y}\bar{z}} \qquad (2a,b)$$

$$F_{y\bar{y}} + F_{z\bar{z}} = 0 \quad . \qquad (2c)$$

Eq.(2b,c) may be incorporated into (2a) by a 'duality transformation' of the coordinate system, analogous to Polyakov's transformation for the chiral model :

$$\begin{aligned} y &\longrightarrow \frac{1}{\sqrt{1+\lambda^2}}\,(y - \lambda\bar{z}) \\ z &\longrightarrow \frac{1}{\sqrt{1+\lambda^2}}\,(z + \lambda\bar{y}) \end{aligned} \qquad (3)$$

This is just an SO(4,C) rotation of C^4, where λ is a complex dimensionless parameter. Under this transformation, the zero-curvature equation

$$0 = F_{yz} \longrightarrow \frac{1}{(1+\lambda^2)} F_{(y-\lambda\bar{z})(z+\lambda\bar{y})} = \frac{1}{1+\lambda^2}\left(F_{yz} + \lambda(F_{y\bar{y}} + F_{z\bar{z}}) + \lambda^2 F_{\bar{y}\bar{z}}\right) = 0 \qquad (4)$$

yielding the linear system [15,16] with (2) as integrability conditions:

$$(\lambda D_y - D_{\bar{z}})H(\lambda) = 0 \qquad (5)$$

$$(\lambda D_z + D_{\bar{y}})H(\lambda) = 0 \quad . \qquad (6)$$

The $(y - \lambda\bar{z})$-$(z + \lambda\bar{y})$ planes on which the curvature (4) vanishes are just the β-planes of twistor geometry.

Equations (2a,b) may be integrated immediately by writing [50,51]

$$A_y = D^{-1}\partial_y D \quad , \quad A_z = D^{-1}\partial_z D \qquad (7)$$

$$A_{\bar{y}} = \bar{D}^{-1}\partial_{\bar{y}}\bar{D} \quad , \quad A_{\bar{z}} = \bar{D}^{-1}\partial_{\bar{z}}\bar{D} \quad , \qquad (8)$$

where $D, \bar{D}$ are elements of the (complexified) gauge group. Gauge transformations correspond to

$$D \longrightarrow D\,G \quad , \qquad \bar{D} \longrightarrow \bar{D}\,G \quad , \tag{9}$$

so that the matrix

$$J = D\,\bar{D}^{-1} \tag{10}$$

is gauge-invariant. In terms of this gauge invariant matrix, (2c) takes the form [51]

$$\partial_{\bar{y}}\left(J^{-1}\partial_y J\right) + \partial_{\bar{z}}\left(J^{-1}\partial_z J\right) = 0 \ ; \tag{11}$$

which displays the important similarity with the two dimensional chiral model field equation. By analogy with the chiral model, we may write down a linear system for which (11) is the integrability condition :

$$\left(\lambda\partial_y - \partial_{\bar{z}} + \lambda J^{-1}\partial_y J\right) Q = 0 \tag{12}$$

$$\left(\lambda\partial_z + \partial_{\bar{y}} + \lambda J^{-1}\partial_z J\right) Q = 0 \quad . \tag{13}$$

This linear system is related to (7,8) by the gauge transformation :

$$\begin{aligned} A_a &\longrightarrow \bar{D} A_a \bar{D}^{-1} + \bar{D}\partial_a \bar{D}^{-1} \ ; \quad a = y, \bar{y}, z, \bar{z} \ , \\ H &\longrightarrow \bar{D} H \equiv Q \quad , \end{aligned} \tag{14}$$

which gauges away $A_{\bar{y}}$ and $A_{\bar{z}}$.

We note that the gauge function in (14) $\bar{D}^{-1} = H(\lambda=0)$.

Equations (11-13) provide, in direct analogy with the chiral model, the non-local currents and non-local symmetries of self-dual gauge fields. The nonlocal currents may be derived [52,53] using a direct generalization of the algorithm of [20] since (11) has the form of a continuity equation, which may be solved in terms of a function $X^{(1)}$ defined by

$$\begin{aligned} \partial_{\bar{z}} X^{(1)} &= J^{-1}\partial_y J = j_y^{(1)} \\ \partial_{\bar{y}} X^{(1)} &= J^{-1}\partial_z J = j_z^{(1)} \end{aligned} \tag{15}$$

Then, writing Q as a power series

$$Q = \sum_{n=0}^{\infty} \lambda^n X^{(n)} \ , \quad X^{(0)} = \mathbb{1} \quad , \tag{16}$$

which we may do if H is analytic around the origin of the complex λ-plane, we clearly obtain the n-th current with components:

$$\begin{aligned} j_y^{(n)} &= \nabla_y X^{(n)} = \left(\partial_y + J^{-1}\partial_y J\right) X^{(n)} = \partial_{\bar{z}} X^{(n-1)} \\ j_z^{(n)} &= \nabla_z X^{(n)} = \left(\partial_z + J^{-1}\partial_z J\right) X^{(n)} = \partial_{\bar{y}} X^{(n-1)} \end{aligned} \tag{17}$$

satisfying the continuity equation

$$\partial_{\bar{y}} j_y^{(n)} + \partial_{\bar{z}} j_z^{(n)} = 0 \quad . \tag{18}$$

Along with these nonlocal continuity equations, we have non-local symmetries of (11). The first three infinitesimal transformations were explicitly written down by Pohlmeyer [52] . As before, they have a structure consistent with the general form [35,54] :

$$\delta J = - J S(\lambda) \tag{19}$$

where
$$S = Q T Q^{-1} = \sum_{n=0}^{\infty} \lambda^n \Lambda^{(n)} \tag{20}$$

satisfies the equations

$$\begin{pmatrix} \lambda \partial_y - \partial_{\bar{z}} \\ \lambda \partial_z + \partial_{\bar{y}} \end{pmatrix} S = - \lambda \begin{pmatrix} [J^{-1}\partial_y J , S] \\ [J^{-1}\partial_z J , S] \end{pmatrix} \quad . \tag{21}$$

The proof that the transformations (19) are symmetries of the linear system (12,13) (or equivalently (21)) is a direct generalization of that given for the chiral model in the previous chapter [54,57] .

Equation (11) is equivalent to (2c) because

$$F_{a\bar{a}} = \bar{D}^{-1} \partial_{\bar{a}} (J^{-1} \partial_a J) \bar{D} \quad ; \quad a = y, z \; . \tag{22}$$

Now, we also have the relation

$$F_{a\bar{a}} = D^{-1} \partial_a (J \partial_{\bar{a}} J^{-1}) D \quad ; \quad a = y, z \; . \tag{22'}$$

So (11) has the equivalent form [51] :

$$\partial_y (J \partial_{\bar{y}} J^{-1}) + \partial_z (J \partial_{\bar{z}} J^{-1}) = 0 \quad , \tag{23}$$

which is the integrability condition for the system :

$$\left(\partial_y - \tfrac{1}{\lambda} \partial_{\bar{z}} - \tfrac{1}{\lambda} J \partial_{\bar{z}} J^{-1} \right) W = 0 \tag{24}$$

$$\left(\partial_z + \tfrac{1}{\lambda} \partial_{\bar{y}} + \tfrac{1}{\lambda} J \partial_{\bar{y}} J^{-1} \right) W = 0 \quad . \tag{25}$$

Because of (22'), these equations are clearly gauge equivalent to the pair of equations, which in addition to (7,8) give the linear system of Atiyah and Ward [17] :

$$(\lambda D_y - D_{\bar{z}}) K(\lambda) = 0 \tag{26}$$

$$(\lambda D_z + D_{\bar{y}}) K(\lambda) = 0 \quad , \tag{27}$$

where $K(\lambda)$ is analytic at $\lambda = \infty$.

Now $K(\lambda = \infty) = D^{-1}$, and

$$W = DK = \sum_{n=0}^{\infty} \lambda^{-n} X^{(-n)} \quad , \quad X^{(0)} = \mathbb{1} \quad . \tag{28}$$

Equations (23,25) together with the expansion (28) and the identification

$$\begin{aligned} J^{(1)}_{\bar{y}} &= J \partial_{\bar{y}} J^{-1} = \partial_z X^{(-1)} \\ J^{(1)}_{\bar{z}} &= J \partial_{\bar{z}} J^{-1} = \partial_y X^{(-1)} \end{aligned}$$

clearly yields another infinite set of continuity equations

$$\partial_y J^{(n)}_{\bar{y}} + \partial_z J^{(n)}_{\bar{z}} = 0 \quad . \tag{29}$$

This may be combined with (18) to give an SO(4,C)-invariant form of the infinite set of continuity equations. However, the set (29) is not independent of (18), since the 'patching matrix'

$$g = K^{-1} H = (W^{-1} D)(\bar{D}^{-1} Q) = W^{-1} J Q \quad ,$$

which determines the vector potential at any point of the null plane [15], satisfies the identity :

$$(\lambda \partial_y - \partial_{\bar{z}}) g = 0 = (\lambda \partial_z + \partial_{\bar{y}}) g \quad . \tag{30}$$

Analogous to (18), we have another set of infinitesimal symmetry transformations [55] corresponding to (29) :

$$\tilde{\delta} J = - R J \quad , \quad R = W T W^{-1} = \sum_{n=0}^{\infty} \lambda^{-n} \Lambda^{(-n)} \quad . \tag{31}$$

The generators of these symmetry transformations, together with those of (19), carry a representation of the infinite dimensional Lie algebra $G \otimes \mathbb{C}(\lambda, \lambda^{-1})$, whose elements are Laurent polynomials in λ with coefficients in the Lie algebra G of the complexified gauge group. The proof of this statement closely follows the analogous one for the chiral model, and has been written down in the literature by Chau et al [55] , who have also considered symmetry transformations corresponding to real gauge potentials, as has Dolan [58] . The latter transformations take the form

$\delta J = - (J S + R J)$, where J is restricted to be hermitian (for real potentials) and R is the hermitian conjugate of S. We note that the former complex transformations are effectively transformations on just two of the four components of the gauge potential, effected by the transformations:

$$\delta D = - R D \quad , \quad \delta \bar{D} = 0 \qquad \text{for (31);}$$

or

$$\delta D = 0 \quad , \quad \delta \bar{D} = S \bar{D} \qquad \text{for (19).}$$

Conversely, the transformations for real gauge potentials correspond to transformations of all four components of A_μ derived from the transformations:

$$\delta D = -RD \quad , \quad \delta \bar{D} = S\bar{D} \quad ; \quad R = S^\dagger \; .$$

We note that all these hidden symmetry transformations correspond to a left-action on $D, \bar{D}$, whereas gauge transformations (10) correspond to a right action.

We conclude with the remark that just as for the chiral model case, the transformations (19) may be expressed recursively in terms of just $(J^{-1}\partial_y J)$ by using only (12) to define the Q in (20). Then, as before, one may construct an exponential form of the function Q (analogous to (2-31)). Similarly, W may be expressed as an exponential series in inverse powers of λ by integrating just equation (24). We also note that the similarity to the chiral model extends to the fact that the lagrangian for eq.(11) :

$$\mathcal{L} = \mathrm{tr}\left\{\partial_y J^{-1}\partial_{\bar{y}} J + \partial_z J^{-1}\partial_{\bar{z}} J\right\} \tag{32}$$

is left invariant up to a total divergence under the transformations (19) with Q satisfying just (12) [35] .

Proof:

$$\begin{aligned}
\delta\mathcal{L} &= \mathrm{tr}\left\{ J^{-1}\partial_y J\,\partial_{\bar{y}} S + J^{-1}\partial_z J\,\partial_{\bar{z}} S + h.c.\right\} \\
&= \mathrm{tr}\left\{ J^{-1}\partial_y J\,\partial_{\bar{y}} S + \lambda J^{-1}\partial_z J\,\partial_y S + \lambda\left[\partial_y(J^{-1}\partial_z J) - \partial_z(J^{-1}\partial_y J)\right] S\right\} + h.c.\text{, using(21a);} \\
&= \mathrm{tr}\left\{ (J^{-1}\partial_y J)(\partial_{\bar{y}} S + \lambda\partial_z S) + \lambda\left[\partial_y(J^{-1}\partial_z J.S) - \partial_z(J^{-1}\partial_y J.S)\right] + h.c.\right\} \\
&= \mathrm{tr}\left\{ -\tfrac{1}{\lambda} Q(\partial_{\bar{z}} - \lambda\partial_y)Q(\partial_{\bar{y}} S + \lambda\partial_z S) \right.\\
&\qquad \left. + \lambda\left[\partial_y(J^{-1}\partial_z J.S) - \partial_z(J^{-1}\partial_y J.S)\right]\right\} + h.c.\text{, using(12);} \\
&= \mathrm{tr}\left\{\lambda\left[\partial_y(Q^{-1}\partial_z Q)T - \partial_z(Q^{-1}\partial_y Q)T + \partial_y(J^{-1}\partial_z J.S) - \partial_z(J^{-1}\partial_y J.S)\right]\right. \\
&\qquad + \tfrac{1}{\lambda}\left[\partial_{\bar{y}}(Q^{-1}\partial_{\bar{z}} Q)T - \partial_{\bar{z}}(Q^{-1}\partial_{\bar{y}} Q)T\right] \\
&\qquad \left. + \left[\partial_z(Q^{-1}\partial_{\bar{z}} Q)T - \partial_{\bar{z}}(Q^{-1}\partial_z Q)T + \partial_y(Q^{-1}\partial_{\bar{y}} Q)T - \partial_{\bar{y}}(Q^{-1}\partial_y Q)T\right]\right\} + h.c.
\end{aligned}$$

The implications of this for self-dual gauge theories are not clear, since (32) describes a non-gauge-invariant theory : the gauge freedom having been used up in going to the manifestly gauge-invariant description in terms of the J-field. However, we may also note that the Yang-Mills lagrangian is itself a total divergence for self-dual fields, since it is equal to the topological density [51] :

$$\begin{aligned}
\mathcal{L} &= \mathrm{tr}\ F^{\mu\nu} F_{\mu\nu} \\
&= \tfrac{1}{2}\ \mathrm{tr}\ \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \\
&= 4\ \mathrm{tr}\left(F_{z\bar{z}} F_{y\bar{y}} + F_{\bar{y}z} F_{y\bar{z}} \right) \\
&= 4\ \mathrm{tr}\Big(\partial_{\bar{z}} \left[(\partial_{\bar{y}} A_y) A_z - (\partial_{\bar{y}} A_z) A_y \right] \\
&\qquad + \partial_{\bar{y}} \left[(\partial_z A_{\bar{z}}) A_y - (\partial_{\bar{z}} A_y) A_z \right] \Big) \quad ,
\end{aligned}$$

where we denote $\quad A_a \equiv J^{-1} \partial_a J$.

Therefore under our infinitesimal hidden symmetry transformations, the full Yang-Mills lagrangian is also a total divergence. However, since the topological number and Yang-Mills action cannot change infinitesimally, both would remain unchanged under the above infinitesimal transformations, unless these transformations gave rise to singular gauge potentials [see also [51, 56, 80]].

3.2 : The functional formulation of 3 dimensional gauge theories.

The path-ordered phase factor is a useful object for the elegant description of many features of gauge fields. In fact it is a natural object to consider since, unlike the field strength, it completely encodes the physical content of gauge theories in the sense that it contains all the gauge-invariant information. There have therefore been many attempts to write the theory solely in terms of these objects [59-61,69]. It was Polyakov [59] who introduced the remarkable idea that every gauge potential $A_\mu(x)$ with values in the Lie algebra of a compact Lie group, corresponds to a chiral field $\Psi(C)$ on the space of curves (loop space), taking values in the holonomy group; and defined by means of the path-ordered integral

$$\Psi(C) = P \exp \int_C A_\mu dx^\mu ; \tag{1}$$

where C is a continuously differentiable oriented closed curve. He showed that if the gauge field satisfies the full Yang-Mills equations, $D_\mu F_{\mu\nu} = 0$, the chiral field $\Psi(C)$ satisfies

$$\frac{\delta F_\mu (C,s)}{\delta x_\mu (s)} = 0 \tag{2}$$

where

$$F_\mu (C,s) = \frac{\delta \Psi(C)}{\delta x^\mu(s)} \Psi^{-1}(C) \tag{3}$$

is the loop space connection form with zero curvature:

$$\delta_\mu(s') F_\nu(s) - \delta_\nu(s) F_\mu(s') + [F_\mu(s), F_\nu(s')] \equiv 0 , \tag{4}$$

where $\delta_\mu(s)$ denotes $\delta/\delta x_\mu(s)$, the functional derivative with respect to the curve $x_\mu(s)$. This identity corresponds to the usual Bianchi identity $\epsilon_{\mu\nu\rho\sigma} [D_\mu, F_{\sigma\rho}] \equiv 0$. The remarkable similarity between two dimensional chiral fields and gauge fields for $d \geqslant 2$ that these equations reveal was exploited by Polyakov to obtain interesting information on the structure of both classical and quantum gauge fields. In particular, he showed that in the case of 2 + 1 dimensions, there exist an infinity of functionally conserved currents, reinforcing the similarity with two-dimensional completely integrable models.

We shall first discuss the possibility of obtaining Polyakov's functionally conserved currents from an action principle. The formal apparatus for the study of functional (path-dependent) fields was developed some time ago by Marshall and Ramond [66], who proved a suitable generalization of Noether's theorem for functional fields.
In the context of loop space fields, Dolan [64a] showed that transformations of the path dependent field induced by transformations of the coordinates of the path change the lagrangian by a total functional divergence and the fifteen corresponding path-dependent Noether currents were constructed. Further, the existence of a higher symmetry of the loop space lagrangian was alluded to. Here we shall show that the reparametrization invariant lagrangian [64a] giving the loop space field equations (2):

$$\mathcal{L} = \operatorname{tr} \int \frac{ds}{\sqrt{x'(s)^2}} \frac{\delta \Psi}{\delta x_\mu(s)} \frac{\delta \Psi^{-1}}{\delta x_\mu(s)} \quad , \mu = 0, 1, 2 , \tag{5}$$

is indeed unchanged up to a total functional derivative under an infinitesimal transformation of the form $\Delta\Psi = \Psi(C+\delta C) - \Psi(C) = -\Psi S[\gamma; x]$ without the use of the equations of motion (2), if $S[\gamma; x]$ satisfies the functional differential equation:

$$(\delta_2 + \gamma t_0 \delta_1 - \gamma t_1 \delta_0) S(\gamma) = -[F_2, S(\gamma)] \quad , \tag{6}$$

where δ_i denotes $\delta/\delta x_i(s)$; and $t_i = x'_i(s) = dx_i(s)/ds$.
Here $S(\gamma)$ is defined to be a power series in negative powers of the parameter γ; and is the generating functional of an infinite set of infinitesimal transformations. We assume that $S(\gamma)$ may be written in the familiar form

$$S(\gamma) = Q(\gamma)\, T\, Q(\gamma)^{-1} \quad , \tag{7}$$

where T is a generator of a constant Lie algebra valued infinitesimal transformation.
Then, from (6), Q( ) satisfies

$$(\delta_2 + F_2 + \gamma t_0 \delta_1 - \gamma t_1 \delta_0) Q = 0 \quad . \tag{8}$$

Now under an infinitesimal transformation $\Delta\Psi = -\Psi S$, the change in the lagrangian density (5) is given by

$$\delta \mathcal{L} = \mathrm{tr} \int \frac{ds}{\sqrt{x'(s)^2}} \quad F_\mu \delta_\mu S(\gamma) \quad . \tag{9}$$

Consider from (8)

$$\mathrm{tr}\ F_2 \delta_2 S = \mathrm{tr}\ F_2 \gamma (t_0 \delta_1 S - t_1 \delta_0 S) \tag{10}$$

and

$$\begin{aligned} \mathrm{tr}\ F_0 \delta_0 S &= \mathrm{tr}\left\{ \frac{t_0}{t_1} F_0 \delta_1 S + \left(\delta_2 (F_0 S) - \delta_0 (F_2 S)\right) \frac{1}{\gamma t_1} + \frac{1}{\gamma t_1} F_2 \delta_0 S \right\} \\ &= \mathrm{tr}\left\{ - F_1 \delta_1 S - \frac{t_2}{t_1} F_2 \delta_1 S + \frac{1}{\gamma t_1}\left(\delta_i (F_j S)\epsilon_{2ij} + F_2 \delta_0 S\right)\right\} \end{aligned} \tag{11}$$

where we have used $t_i F_i = 0$; an identity which is easily obtained by considering the variation of ψ by an infinitesimal vector field [60-63] :

$$\frac{\delta}{\delta x_\mu(s)} \psi(s,C) = F_{\mu\sigma} t_\sigma \psi(s,C) \quad ,$$

where a trace over the base point of the contour is implicit in our notation. Thus

$$\frac{\delta \psi(s,C)}{\delta x_\mu(s)} \psi^{-1}(s,C) = F_{\mu\sigma} t_\sigma = F_\mu \quad , \tag{12}$$

which immediately gives $t_\mu F_\mu = 0$ because of the antisymmetry of the field strength tensor $F_{\mu\sigma}$. Equivalently, parametrization invariance yields $0 = \frac{d}{ds}\psi(s,C) = \frac{dx_\mu(s)}{ds}\frac{\delta}{\delta x_\mu(s)} \psi(s,C)$.

Now from (10,11) we obtain

$$\begin{aligned} \int \frac{ds}{\sqrt{x'^2(s)}} F_i \delta_i S &= \mathrm{tr} \int \frac{ds}{\sqrt{x'^2(s)}} \left\{ F_2 \left(\gamma t_0 \delta_1 S - \gamma t_1 \delta_0 S - \frac{t_2}{t_1} \delta_1 S + \frac{1}{\gamma t_1} \delta_0 S \right) \right. \\ &\qquad \left. + \frac{1}{\gamma t_1} \delta_i (F_j S) \epsilon_{1ij} \right\} \\ &= \mathrm{tr} \int \frac{ds}{\sqrt{x'^2(s)}} \left\{ Q (\delta_2 + \gamma t_0 \delta_1 - \gamma t_1 \delta_0) Q^{-1} \left(\left(\gamma t_0 - \frac{t_2}{t_1}\right)\delta_1 S + \left(\frac{1}{\gamma t_1} - \gamma t_1\right)\delta_0 S \right) \right. \\ &\qquad \left. + \frac{1}{\gamma t_1} \delta_i (F_j S) \epsilon_{1ij} \right\} \quad \text{, using (8);} \\ &= \mathrm{tr} \int \frac{ds}{\sqrt{x'^2(s)}} \left\{ \gamma t_j \epsilon_{jik} \delta_i (\delta_k Q^{-1}.Q) + \frac{1}{\gamma t_1} \epsilon_{1ij} \delta_i (\delta_j Q^{-1}.Q) \right. \\ &\qquad \left. + \frac{1}{\gamma t_1} \delta_i (F_j S) \epsilon_{1ij} + \delta_i \left\{ \frac{t_0}{t_1} \epsilon_{ij2} \delta_j Q^{-1}.Q - \frac{t_2}{t_1} \epsilon_{0ij} \delta_j Q^{-1}.Q \right\}\right\} T \quad , \end{aligned}$$

using (7), the cyclic property of the trace, and the fact that in the integrand $\delta'(0) = 0$ (c.f. [66]).

Now the antisymmetry of the integrand allows us to finally write

$$\delta \mathcal{L} = tr \int ds \frac{\delta}{\delta x_i(s)} \left\{ \frac{1}{\sqrt{x'^2(s)}} \left[\gamma t_j \epsilon_{jik} \delta_k Q^{-1} . Q + \frac{1}{\gamma t_1} \epsilon_{1ij} (\delta_j Q^{-1} . Q + F_j S) + \frac{t_0}{t_1} \epsilon_{ij2} \delta_j Q^{-1} . Q - \frac{t_2}{t_1} \epsilon_{0ij} \delta_j Q^{-1} . Q \right] T \right\} . \quad (13)$$

We may write the defining relation (8) for $Q(\gamma)$ as

$$Q (\delta_2 + \gamma \epsilon_{2\nu\rho} t_\nu \delta_\rho) Q^{-1} = F_2 = \delta_2 \Psi . \Psi^{-1} . \quad (14)$$

Now, analogously with the case of the chiral model, if we set

$$\delta_2 \Psi . \Psi^{-1} = \epsilon_{2ij} \delta_j X^{(1)} t_i \quad (15)$$

and $$Q = \lim_{N \to \infty} Q_N ; \quad Q_N = e^{X_{(1)} \gamma^{-1}} e^{X_{(2)} \gamma^{-2}} \ldots\ldots\ldots\ldots e^{X_{(N)} \gamma^{-N}} \quad (16)$$

the coefficient of γ^0 automatically satisfies (14), allowing, recursively, an explicit construction of the coefficients of γ^{-n}. We have thus shown that a power series solution of (6) exists. Now we note that when (2) is satisfied, the integrand in (9) is also a total functional divergence:

$$\delta \mathcal{L} = tr \int ds \ \delta_\mu \left(\frac{1}{\sqrt{x'(s)^2}} F_\mu S(\gamma) \right) . \quad (17)$$

From(13,17) we may deduce that an infinite number of functionally conserved Noether currents exist, since the integral over the parametrization is over a compact domain for closed loops. We note that all this is valid as long as the path is regular and has no crossing points; this guarantees that end-point terms do not contribute.

We note that (6) is just one component of the system of equations

$$(\delta_i + \gamma t_j \epsilon_{ijk} \delta_k) S = -[F_i , S] ; \quad i = 0, 1, 2 , \quad (18)$$

first suggested by Polyakov [59] as the linear system for (2-4). The integrability conditions for these equations were checked in detail by Dolan [64b]. Similarly, (8) is a component of the equivalent linear system

$$(\delta_i + F_i + \gamma t_j \epsilon_{ijk} \delta_k) Q = 0 , \quad (19)$$

which has a form which immediately yields itself to the algorithm of [20] for the construction of the infinity of continuity equations [59,65].

The parametrization invariance of $\Psi(s, C)$ implies that $t_i \delta_i Q = 0$ (since $Q(\gamma = 0) = \Psi^{-1}$) in addition to $t_i F_i = 0$. Therefore, the component of (19) in the direction of t_i is an identity; leaving only the two orthogonal components as linearly independent equations. The integrability conditions for (19) may therefore be considered to be the conditions for the vanishing of the curvature on the two dimensional surface orthogonal to t_i . This is the clue to the integrability of three dimensional loop space fields, since the equations of motion are seen to be those of an effectively two dimensional theory. We shall discuss this in further detail in the next section.

We may proceed to show that the symmetry transformation $\Delta\Psi = -\Psi S$, leaves the linear system invariant. Analogously with the two dimensional case, we take the variation of Q due to the infinitesimal transformation $\Delta(\gamma)\Psi = -\Psi S(\gamma)$ to be :

$$\Delta(\beta)\, Q(\gamma) = \frac{\beta}{\beta-\gamma}\left(S(\beta) - S(\gamma)\right) Q(\gamma) .$$

Using this it is easy to show that the linear system (19) is symmetric under the transformation $\Psi \to \Psi + \Delta\Psi$; proving that this parametric symmetry is a symmetry of the equations of motion. We may, in a similar fashion, obtain another set of infinitely many symmetries; generated by the transformation $\Delta(\gamma)\Psi = -\Psi R(\gamma)$ where R is a power series in positive powers of γ , satisfying the system of equations :

$$\left(\delta_i + \frac{1}{\gamma} t_j \epsilon_{ijk} \delta_k\right) R = -\left[F_i , R\right] .$$

The generators of these two infinite sets of symmetries form the loop algebra $G \otimes \mathbb{R}[\gamma, \gamma^{-1}]$, as may be demonstrated following the analogous discussion of chapter 2.

We remark that if we denote $\tilde{\Psi} = \Psi + \Delta\Psi$, then under the symmetry transformation $\Delta\Psi = -\Psi S$, the change in F_i is

$$\begin{aligned}
\Delta F_i &= \delta_i S + [F_i , S] \\
&= -\gamma t_j \epsilon_{ijk} \delta_k S \qquad \text{, using (18);} \\
&= \gamma t_j \epsilon_{ijk} \delta_k (\Psi^{-1}\tilde{\Psi}) \qquad \text{, since } S = -\Psi^{-1}\tilde{\Psi} .
\end{aligned}$$

Thus we have the equations

$$\delta_i \tilde{\Psi} . \tilde{\Psi}^{-1} - \delta_i \Psi . \Psi^{-1} = \gamma t_j \epsilon_{ijk} \delta_k (\Psi^{-1} \tilde{\Psi})$$

relating two solutions Ψ and $\tilde{\Psi}$ of (2). These equations are analogous to the Backlund transformation for the chiral model exhibited in [39].

To conclude this section, we note that the loop space symmetry described here does not seem to have any direct implication for the existence of higher symmetries of ordinary three dimensional gauge fields. This is not surprising since the lagrangian (5) is not equal to the ordinary Yang-Mills lagrangian; there exist loop space fields which do not correspond to gauge fields. (The topological criteria which distinguish loop space fields which do not correspond to usual gauge fields are discussed in [67], where it is shown that the loop space fields corresponding to usual gauge fields are 'more continuous' than the others, in the sense that if we have a sequence of curves C_n (parametrized by $x_n^\mu(s)$) tending to the curve $C(x^\mu(s))$, then $x_n^\mu(s)$ converges uniformly to $x^\mu(s)$ and the sequence $t_n^\mu = dx_n^\mu(s)/ds$ is uniformly bounded. The authors of [67] claim that a field $\Psi(C)$ corresponding to a usual gauge field is continuous in this topology of the space of curves; i.e. if C_n tends to C then $\Psi(C_n)$ tends to $\Psi(C)$.) However, functionally conserved currents do impose (via higher Ward identities) strong constraints on the interaction between closed gauge strings [59], which afterall are the objects expected to play the role of elementary excitations in the confining phase of the theory [68].

3.3 : Some integrable chiral fields in three dimensions.

(i) We have seen that the clue to the integrability of the functional formulation of three dimensional gauge theories is the fact that parametrization invariance constrains the theory to a two dimensional subspace of space-time. In order to understand this effective dimensional reduction we choose to simulate the loop space linear system

$$(\delta_i + F_i + \gamma t_j \epsilon_{ijk} \delta_k) Q = 0$$

by the ordinary space linear equations

$$(\delta_{ik} + \lambda \epsilon_{ijk} v_j) \partial_k \psi = - A_i \psi \;, \quad i = 1,2,3 \;, \tag{1}$$

where the constraint

$$v_i A_i = 0 \tag{2}$$

implies that $v_i \partial_i \psi = 0$. We choose v_i to be a unit vector: $v^2 = 1$. Eq.(1), where A_i satisfies (2), has the equivalent form

$$\left[\partial_\ell + \frac{1}{1+\lambda^2} (\delta_{\ell i} - \lambda \epsilon_{\ell m i} v_m) A_i \right] \psi = 0 \tag{3}$$

since

$$(\delta_{ik} + \lambda \epsilon_{ijk} v_j)(\delta_{k\ell} - \lambda \epsilon_{km\ell} v_m + \lambda^2 v_k v_\ell) = \delta_{i\ell} (1+\lambda^2) .$$

The consistency conditions for (3) are

$$F_{\ell m} \equiv \partial_\ell A_m - \partial_m A_\ell + [A_\ell, A_m] = 0 \tag{4a}$$

and

$$\epsilon_{\ell q p} \partial_m (v_q A_p) - \epsilon_{mni} \partial_\ell (v_n A_i) = 0 \;; \tag{4b}$$

or equivalently,

$$A_i = g^{-1} \partial_i g \quad , \text{ a pure-gauge; and}$$

$$\partial_i (v_j A_i - v_i A_j) = 0 \quad . \tag{5}$$

We note that the linear system (3) alludes to a 'duality transformation':

$$A_i \longrightarrow \frac{1}{1+\lambda^2} (A_i - \lambda \epsilon_{ijk} v_j A_k) \tag{6}$$

mapping pure-gauge fields to solutions of (5).

(ii) The linear system (1) allows the construction of an infinite set of conserved currents. To explicitly demonstrate this, we first solve (4) by writing

$$A_k = - \epsilon_{kmn} v_m \partial_n \varphi^{(1)} \quad , \tag{7}$$

which we take to be the first current: $J_k^{(1)} = A_k$.

Then, $\partial_k J_k^{(1)} = 0$ if $\epsilon_{kmn} \partial_k v_m = 0$. (8)

We note that unlike the case of the two dimensional chiral model, this first continuity equation is not the field equation. Now, if we assume that in (1) we may write

$$\Psi = \sum_{n=0}^{\infty} \lambda^{-n} \varphi^{(n)} \quad , \quad \varphi^{(0)} = \mathbb{1} \quad ,$$

then (7) is clearly the λ-independent piece of (1); and $\varphi^{(n)}$ satisfies the recurrence relation

$$\epsilon_{kmn} v_m \partial_n \varphi^{(n)} = - \left(\partial_k + A_k \right) \varphi^{(n-1)} \quad . \tag{9}$$

The n-th conserved current is then obtained from $\varphi^{(n)}$:

$$J_k^{(n)} = \epsilon_{kmn} v_m \partial_n \varphi^{(n)} \quad . \tag{10}$$

We observe that $J_k^{(n)}$ not only satisfies $\partial_k J_k^{(n)} = 0$, but is also a solution of the equations of motion since $\partial_\ell \left(v_\ell J_k^{(n)} - v_k J_\ell^{(n)} \right) = 0$, which may be verified by repeated use of (9). This, however, is not the only solution generating method available to these equations, since the linear system (1) clearly has a form which allows the construction of solutions using the Riemann-Hilbert transform (c.f. eq.(1-8)).

(iii) The consistency conditions (4) superficially represent the vanishing of the curvature of the connection in (3) on a 3-plane. However, because of the condition (2), the system (3) only contains two independent equations for Ψ ; the component of (3) in the direction of v_i being trivially satisfied. The consistency conditions therefore have the conventional interpretation of the vanishing of a curvature on a 2-plane; which in the present case is the two dimensional surface orthogonal to v_i . Our theory is thus effectively a two dimensional one. We note that if A_i satisfies (2,5), it also satisfies

$$v_j \partial_i \left(v_i A_j - v_j A_i \right) = v_j v_i \partial_i A_j - \partial_i A_i = 0 \tag{11}$$

and

$v_i F_{ij} = 0$ implies that

$$v_i \partial_i A_j = - \partial_j v_i \,.\, A_i \tag{12}$$

So equation (5) may be rewritten as an algebraic equation for A_i :

$$0 = \partial_i v_i A_j - (\partial_j v_i + \partial_i v_j) A_i - v_j \partial_i A_i$$
$$= (\partial_k v_k \delta_{ij} - (\partial_j v_i + \partial_i v_j) + v_j \partial_\ell v_i v_\ell) A_i \quad , \text{ using (11);}$$

explicitly demonstrating that the components of A_i are not linearly independent. The matrix acting on A_i may clearly be written as a symmetric matrix:

$$(\partial_k v_k \delta_{ij} - (\partial_j v_i + \partial_i v_j) + \partial_\ell (v_j v_i) v_\ell) A_i = 0 \quad ;$$

and as a traceless matrix:

$$\left(\partial_k v_k (\delta_{ij} - v_i v_j) - (\partial_j v_i + \partial_i v_j) + \partial_\ell (v_j v_i) v_\ell\right) A_i = 0 \, . \tag{13}$$

Now, if $\partial_k v_k \neq 0$, we obtain a linear relation between the three components of A_i :

$$A_j = \frac{1}{(\partial_k v_k)} \left\{ (\partial_j v_i + \partial_i v_j) - \partial_\ell (v_j v_i) v_\ell \right\} A_i \; , \tag{14}$$

which is completely equivalent to the equations of motion. We now observe that (11) may be written

$$(\delta_{ij} - v_i v_j) \partial_i A_j = 0 \quad , \tag{15}$$

where the projection tensor

$$P_{ij} = \delta_{ij} - v_i v_j$$

which has the properties

$$P_{ij} P_{jk} = P_{ik} \quad , \quad P_{ij} v_j = 0 \quad ,$$

may be used to decompose every tensor into its components parallel or normal to v_i . P_{ij} is clearly the degenerate metric on the 2-surface orthogonal to v_i . We have already alluded two the fact that this theory is actually two dimensional. If this is true, v_i needs to be interpreted as a Killing vector field generating an isometry of the orthogonal 2-surface. In other words, we need to interpret translations in the direction of v_i as either mappings of the 2-space onto itself, or as a motion (like a rotation) of the space which does not alter the metric P_{ij} . We consider infinitesimal transformations $x_i \rightarrow x_i + v_i$.

The condition for $v_i(x)$ itself to be invariant under this transformation is clearly

$$\mathcal{L}_v v_i = v_j \partial_j v_i = 0 \quad . \tag{16}$$

The Lie derivative of the metric, which we multiply by a conformal scale f, is given by

$$\mathcal{L}_v\left(f(\delta_{ij}-v_iv_j)\right) = v_k\partial_k\left(f(\delta_{ij}-v_iv_j)\right) + f\,\partial_i v_k(\delta_{kj}-v_kv_j) + f\,\partial_j v_k(\delta_{ik}-v_iv_k)$$

$$= v_k\partial_k f\cdot(\delta_{ij}-v_iv_j) + f(\partial_i v_j+\partial_j v_i) - f v_k\partial_k(v_iv_j). \quad (17)$$

The last term vanishes by (16). Moreover, if the conformal factor f satisfies

$$\partial_k(v_k f) = 0$$

i.e. $f = \epsilon_{klm} v_k \partial_l u_m$, for some u_m,

which implies that

$$v_k\partial_k \ln f = -\partial_k v_k \quad ,$$

then (17) vanishes if v_i satisfies

$$\partial_i v_j + \partial_j v_i = \tfrac{1}{2}\, g_{ij}\,\partial_k v_k \quad ; \quad \tfrac{1}{2} g_{ij} = f(v_iv_j - \delta_{ij}) \; ; \quad (18)$$

i.e. the conformal Killing equation (see e.g. [83]).

We also have

$$\delta A_i = \mathcal{L}_v A_i = v_j\partial_j A_i + A_j\partial_i v_j = 0, \text{ since } v\cdot A = 0,\ F_{ij} = 0 .$$

Since the pure-gauge A_i is left invariant, the transformation $x_i \to x_i + v_i$ is clearly a gauge-covariant conformal transformation (c.f. [83]).

We note that (16) together with $v\cdot A = 0$, implies that (15) is equivalent to

$$\partial_i A_i = 0 \quad , \quad (19)$$

the chiral field equation. The condition (16), which is necessary for (19) to be the equation of motion, is precisely analogous to the condition implied by parametrization invariance in the case of the loop-space chiral equations.

We now consider the action

$$S = \int d^3x \;\; tr \;\; F_{ij}\, T_{ij} \quad (20)$$

where T_{ij} is an antisymmetric tensor field satisfying $A_iT_{ij} = 0$.

Varying A_i gives the equation of motion $\partial_i T_{ij} = 0$,

and varying T_{ij} yields $F_{ij} = 0$; equations similar to those under consideration (4a,5). We note that if we choose a particular solution to (16,18): $v_i = x_i/r$, (a choice which is similar to the loop tangent vector $t_i = dx_i(s)/ds$), then (20) may be identified with the chiral-field lagrangian $\int d^3x\, A_i A_i$ if $T_{ij}(x) = \int_1^\infty d\beta\, \beta\, (x_i A_j(\beta x) - x_j A_i(\beta x))$.

The proof follows [84] . The condition $v \cdot A = 0$ is clearly just the coordinate gauge condition $x \cdot A = 0$, which implies that

$$x_i F_{ij} = (1 + x_i \partial_i) A_j$$

Rescaling: $x_i \longrightarrow \alpha x_i$, we find that

$$\alpha x_i F_{ij}(\alpha x) = \frac{d}{d\alpha}\left[\alpha A_j(\alpha x)\right]$$

$$\therefore \quad A_j(x) = \int_0^1 d\alpha \quad \alpha x_i F_{ij}(\alpha x) \quad .$$

Now [84]

$$\begin{aligned}
\int d^3x\, A_i A_i &= \int d^3x \int_0^1 d\alpha \quad \alpha x_k F_{ki}(\alpha x) A_i(x) \\
&= \int d^3x \int d^3y \int_0^1 d\alpha \quad y_k F_{ki}(y)\, \delta(y - \alpha x)\, A_i(x) \\
&= \int d^3y \quad y_k F_{ki}(y) \int_0^1 d\alpha \quad \alpha^{-3} A_i(y/\alpha) \\
&= \frac{1}{2} \int d^3y \int_1^\infty d\beta\, \beta\, F_{ij}(y)\, (y_i A_j(\beta y) - y_j A_i(\beta y)) \; ; \quad \beta = 1/\alpha \; , \\
&= \int d^3y \quad F_{ij}(y) T_{ij}(y) \qquad \text{i.e. } (20).
\end{aligned}$$

(iv) To conclude, we note that the Ernst equation for stationary axisymmetric gravitational fields may be rewritten in terms of the type of chiral fields we have considered in this section (c.f. [82,85]). We may immediately obtain equations for such cylindrically symmetric chiral fields by making the choice : $v = \frac{1}{\rho}(x_2 , -x_1 , 0)$, $\rho = x_1^2 + x_2^2$, $(x_i \in \mathbb{R}^3, i = 1,2,3)$. Moreover, our discussion may be extended to an n-dimensional space, as long as we reduce the effective dimensionality of the problem to two by introducing (n - 2) commuting Killing-vector fields. For example, in four dimensions we may start with the linear system

$$(\delta_{im} + \lambda\, \epsilon_{ijkm} v_j u_k)\, \partial_m \Psi = -A_i \Psi \; ; \; v^2 = 1 = u^2 , \; v \cdot u = 0 , \; v \cdot A = 0 = u \cdot A .$$

The consistency conditions for this system correspond to equations for chiral fields over a 4-dimensional space-time admitting two Killing vectors.

Chapter 4: Completely integrable sectors of supersymmetric gauge theories.

Supersymmetric gauge theories have a natural formulation in superspace, which we take to be parametrized by $y = (x^{\alpha\dot{\beta}}, \theta_\alpha^{\ s}, \bar{\theta}^{\dot{\beta} t})$, $\alpha, \dot{\alpha} = 1, 2$; $s, t = 1, \ldots\ldots, N$; where $x^{\alpha\dot{\beta}} = x^a \sigma_a^{\ \alpha\dot{\beta}}$. The gauge-covariant derivatives may be written

$$\nabla_\alpha^{\ s} = D_\alpha^{\ s} + A_\alpha^{\ s} \quad , \quad \bar{\nabla}_{\dot{\beta} t} = \bar{D}_{\dot{\beta} t} + A_{\dot{\beta} t} \ , \quad \nabla_{\alpha\dot{\beta}} = \partial_{\alpha\dot{\beta}} + A_{\alpha\dot{\beta}} \quad , \text{ where}$$

$$D_\alpha^{\ s} = \frac{\partial}{\partial \theta_s^{\ \alpha}} + i \bar{\theta}^{\dot{\beta} s} \partial_{\alpha\dot{\beta}} \ , \ \bar{D}_{\dot{\beta} t} = - \frac{\partial}{\partial \bar{\theta}^{\dot{\beta} t}} - i \theta_t^{\ \alpha} \partial_{\alpha\dot{\beta}} \ , \ \partial_{\alpha\dot{\beta}} = \frac{\partial}{\partial x^{\alpha\dot{\beta}}}$$

realise the supersymmetry algebra

$$\{ D_\alpha^{\ s}, D_\beta^{\ t} \} = 0 = \{ \bar{D}_{\dot{\beta} s}, \bar{D}_{\dot{\alpha} t} \} \quad , \quad \{ D_\alpha^{\ s}, \bar{D}_{\dot{\beta} t} \} = - 2 i \delta^s_t \ \partial_{\alpha\dot{\beta}} \ ;$$

and $A_{\alpha\dot{\beta}}$, $A_\alpha^{\ s}$, $A_{\dot{\beta} t}$ are the Lie algebra valued Yang-Mills superfield potentials. The above gauge-covariant derivatives yield the superfield curvatures of the theory [70]

$$\{ \nabla_\alpha^{\ s}, \nabla_\beta^{\ t} \} = F^{st}_{\alpha\beta} \quad , \quad \{ \bar{\nabla}_{\dot{\alpha} s} \ , \bar{\nabla}_{\dot{\beta} t} \} = F_{\dot{\alpha} s, \dot{\beta} t} \tag{1}$$

$$[\nabla_\mu , \nabla_\alpha^{\ s}] = F^{\ s}_{\mu\alpha} \quad , \quad [\nabla_\mu , \bar{\nabla}_{\dot{\beta} t}] = F_{\mu, \dot{\beta} t} \ , \ [\nabla_\mu , \nabla_\nu] = F_{\mu\nu} \tag{2}$$

and

$$\begin{aligned} \{ \nabla_\alpha^{\ s}, \bar{\nabla}_{\dot{\beta} t} \} &= \{ D_\alpha^{\ s}, \bar{D}_{\dot{\beta} t} \} + D_\alpha^{\ s} A_{\dot{\beta} t} + \bar{D}_{\dot{\beta} t} A_\alpha^{\ s} + \{ A_\alpha^{\ s} , A_{\dot{\beta} t} \} \\ &= F^{\ s}_{\alpha, \dot{\beta} t} \quad - 2 i \ \nabla_{\alpha\dot{\beta}} \ \delta^s_t \end{aligned} \tag{3}$$

since $$F^{\ s}_{\alpha, \dot{\beta} t} = D_\alpha^{\ s} A_{\dot{\beta} t} + \bar{D}_{\dot{\beta} t} A_\alpha^{\ s} + \{ A_\alpha^{\ s} , A_{\dot{\beta} t} \} + 2 i \ A_{\alpha\dot{\beta}} \quad . \tag{4}$$

The latter relation determines the vector potential $A_{\alpha\dot{\beta}}$ in terms of the spinor ones, so the theory in terms of the above six curvature forms is highly redundant, allowing the imposition of extra constraints amongst the curvatures, which do not put the theory on-shell. For $N = 1, 2$, the usual [70] constraint equations are

$$\begin{gathered} F^{st}_{\alpha\beta} + F^{ts}_{\alpha\beta} = 0 = F_{\dot{\alpha} s, \dot{\beta} t} + F_{\dot{\alpha} t, \dot{\beta} s} \\ F^{\ s}_{\alpha, \dot{\beta} t} = 0 \quad . \end{gathered} \tag{5}$$

It is known [71,72] that for the maximally extended cases of $N = 3, 4$ (these being equivalent), these equations put the theory on-shell (without, however, trivializing the theory). Therefore, for $N = 3, 4$, the equations (5) are analogous to the self-duality equations of ordinary gauge theories; they provide a non-trivial on-shell sector of the theory in terms of algebraic relations amongst the components of the gauge curvature form.

Recently, Volovich [73] wrote down a set of linear equations for the spinor connection A^s_α, $A_{\dot\beta t}$, which have equations (5) as consistency conditions. In this chapter, we first integrate a subset of the equations in (5) by introducing two independent gauge functions. The remaining equations are then rewritten in a manifestly gauge-invariant fashion in terms of these functions; and following the approach of Volovich, a linear system for them is written down. This reformulated linear system may be used to construct an infinity of continuity equations for supersymmetric gauge theories, since it yields itself to a modified version of the algorithm of [20], which we discussed in chapter 2. The formulation given here is very similar to; and is suggested by, the manifestly gauge-invariant formulation of self-dual gauge fields [50,51]; the so-called J-formulation of chapter 3. We also show that in the sector of (5) describing self-dual gauge fields [74], our formulation is equivalent to; and is thus a consistent generalization of, the J-formulation.

We choose to solve the following subset of equations in (5):

$$F^{st}_{11} = 0 = F^{st}_{22} \tag{6}$$

$$F^{s}_{1,\dot i t} = 0 = F^{s}_{2,\dot 2 t} \tag{7}$$

by writing

$$A^s_1 = g^{-1} D^s_1 g \quad , \quad A_{\dot 1 t} = g^{-1} \bar D_{\dot 1 t}\, g \tag{8}$$

$$A^s_2 = h^{-1} D^s_2 h \quad , \quad A_{\dot 2 t} = h^{-1} \bar D_{\dot 2 t}\, h \quad , \tag{9}$$

where the superfields g and h are arbitrary elements of the gauge group; and they may be expressed in terms of Lie algebra valued prepotentials conventionally used to solve constraint equations by relations of the form

$$\nabla_1 = D_1 + g^{-1} D_1 g = e^{-U} D_1 e^{U}$$

$$\nabla_2 = D_2 + h^{-1} D_2 h = e^{-V} D_2 e^{V} \quad , \text{ etc.,}$$

where U and V are two independent prepotentials.

We note that because of the definition (4), the relations (7-9) imply a pure-gauge form for two of the vector potentials:

$$A_{1i} = g^{-1}\partial_{1i} g \quad , \quad A_{2\dot{i}} = h^{-1}\partial_{2\dot{2}} h$$

leaving the other two, $A_{1\dot{2}}$ and A_{2i}, undetermined. The remaining constraint equations are:

$$F_{12}^{st} + F_{12}^{ts} = 0 \tag{10}$$

$$F_{is,\dot{2}t} + F_{it,\dot{2}s} = 0 \tag{11}$$

$$F^{s}_{1,\dot{2}t} = 0 = F^{s}_{2,it} \quad . \tag{12}$$

We now note that since gauge transformations correspond to a right action of the gauge group on g and h, the matrix $B = g\,h^{-1}$ is manifestly gauge-invariant; and in terms of it, we may write

$$F_{12}^{st} = g^{-1}\{ D_1^s (B D_2^t B^{-1})\} g \tag{13}$$

$$F_{is,\dot{2}t} = g^{-1}\{ \bar{D}_{is} (B \bar{D}_{\dot{2}t} B^{-1})\} g \tag{14}$$

$$F^{s}_{1,\dot{2}t} = g^{-1}\{ D_1^s (B \bar{D}_{\dot{2}t} B^{-1}) + 2i\,\delta^s_t \nabla_{1\dot{2}} \} g \tag{15a}$$

$$F^{s}_{2,it} = g^{-1}\{ D_{it} (B D_2^s B^{-1}) + 2i\,\delta^s_t \nabla_{2i} \} g \quad . \tag{15b}$$

We may therefore rotate (10-12) by the transformation which takes the pure-gauge potentials A_1^s, A_{it} and A_{1i} to zero; obtaining a set of equations for the gauge-invariant field B, equivalent to equations (10-12):

$$D_1^s (B D_2^t B^{-1}) + D_1^t (B D_2^s B^{-1}) = 0 \tag{16}$$

$$\bar{D}_{is} (B \bar{D}_{\dot{2}t} B^{-1}) + \bar{D}_{it} (B \bar{D}_{\dot{2}s} B^{-1}) = 0 \tag{17}$$

$$D_1^s (B \bar{D}_{\dot{2}t} B^{-1}) + \delta^s_t\, 2i\, g \nabla_{1\dot{2}} g^{-1} = 0 \tag{18}$$

$$\bar{D}_{is} (B D_2^t B^{-1}) + \delta^t_s\, 2i\, g \nabla_{2i} g^{-1} = 0 \quad . \tag{19}$$

We observe that equations (16-19) are consistency conditions for the system of equations:

$$L^s \Psi = (D_1^s + \lambda D_2^s + \lambda B D_2^s B^{-1}) \Psi = 0 \tag{20}$$

$$M_t \Psi = (\bar{D}_{\dot{2}t} + B \bar{D}_{\dot{2}t} B^{-1} + \lambda^{-2} \bar{D}_{it}) \Psi = 0 \tag{21}$$

$$N \Psi = \{ (\partial_{1\dot{2}} + g \nabla_{1\dot{2}} g^{-1}) + \lambda (\partial_{2\dot{2}} + B \partial_{2\dot{2}} B^{-1}) + \lambda^{-2} \partial_{1i} + \lambda^{-1} (\partial_{2i} + g \nabla_{2i} g^{-1}) \} \Psi = 0 \tag{22}$$

since (20-22) result immediately from the algebra:

$$\{ L^s, L^t \} = 0 = \{ M_s, M_t \} \; ; \; \{ L^s, M_t \} = -2i\,\delta^s_t N \; . \tag{23}$$

The linear system (20-22) may be obtained from the one given by Volovich, which we choose to write as:

$$X^s \psi = (\nabla_1^s + \lambda \nabla_2^s) \psi = 0 \tag{24}$$

$$Y_t \psi = (\bar{\nabla}_{\dot{2}t} + \lambda^{-2} \bar{\nabla}_{it}) \psi = 0 \tag{25}$$

$$Z\Psi = (\nabla_{1\dot{2}} + \lambda\nabla_{2\dot{2}} + \lambda^{-2}\nabla_{1\dot{1}} + \lambda^{-1}\nabla_{2\dot{1}})\Psi = 0 \tag{26}$$

(where X^s, Y_t, Z satisfy the algebra

$$\{X^s, X^t\} = 0 = \{Y_s, Y_t\}, \quad \{X^s, Y_t\} = -2i\delta^s_t Z$$

if the constraints (5) are satisfied), by performing the above gauge transformation; since the two systems are related thus:

$$X\Psi = g^{-1}L\Psi, \quad Y\Psi = g^{-1}M\Psi, \quad Z\Psi = g^{-1}N\Psi$$

where $\Psi = g\Psi$ and the gauge function is given by

$$g^{-1} = \Psi(\lambda = 0), \tag{27}$$

which is consistent with the normalization: $\Psi(\lambda = 0) = \mathbb{1}$.

We note that if instead of (27) we choose to rotate (10-12) by the function which transforms $A_1, A_{\dot{j}}, A_{2\dot{i}}$ to zero, i.e.

$$h^{-1} = \varphi(\lambda = \infty) \tag{28}$$

where $\varphi(\lambda)$ also satisfies (24-26), then, since we have relations of the form

$$F^{st}_{12} = h^{-1}\{D^s_2(B^{-1}D^t_1 B)\}h, \tag{29}$$

corresponding to (13-15), eqs.(16-19) have the equivalent form

$$D^s_2(B^{-1}D^t_1 B) + D^t_2(B^{-1}D^s_1 B) = 0 \tag{16'}$$

$$\overline{D}_{\dot{2}s}(B^{-1}\overline{D}_{\dot{i}t}B) + \overline{D}_{\dot{2}t}(B^{-1}\overline{D}_{\dot{i}s}B) = 0 \tag{17'}$$

$$\overline{D}_{\dot{2}s}(B^{-1}D^t_1 B) + \delta^t_s\, 2i\, h\nabla_{1\dot{2}}h^{-1} = 0 \tag{18'}$$

$$D^s_2(B^{-1}\overline{D}_{\dot{i}t}B) + \delta^s_t\, 2i\, h\nabla_{2\dot{i}}h^{-1} = 0, \tag{19'}$$

for which we have the linear system

$$L'^s\Phi = (D^s_2 + \lambda^{-1}D^s_1 + \lambda^{-1}B^{-1}D^s_1 B)\Phi = 0 \tag{20'}$$

$$M'_t\Phi = (\lambda^2\overline{D}_{\dot{2}t} + \overline{D}_{\dot{i}t} + B^{-1}\overline{D}_{\dot{i}t}B)\Phi = 0 \tag{21'}$$

$$N'\Phi = \{\lambda(\partial_{1\dot{2}} + h\nabla_{1\dot{2}}h^{-1}) + \lambda^2\partial_{2\dot{2}} + \lambda^{-1}(\partial_{1\dot{1}} + B^{-1}\partial_{1\dot{1}}B) + (\partial_{2\dot{1}} + h\nabla_{2\dot{1}}h^{-1})\}\Phi = 0, \tag{22'}$$

where $\Phi = h\varphi$, $\Phi(\lambda=\infty) = \mathbb{1}$; and L', M', N' also realise the algebra (23) as a consequence of (16'-19').

We now proceed to construct an infinite set of continuity equations. For explicitness we choose to do this for the N = 3 theory. Our discussion, however, is valid for any N, since in what follows one may delete all terms with the index s, t = 3 and then those with s, t = 2 to successively reduce the theory to $N = 2$ and then to $N = 1$. Further, the generalization to the manifestly $N = 4$ theory is obvious. We note that (16,17) may be solved in terms of two functions of the superspace variables, $X^{(1)}$ and $X^{(2)}$, defined by the relations

$$B D_2^s B^{-1} = D_1^s X^{(1)} \tag{29}$$

$$B \bar{D}_{\dot{2}t} B^{-1} = \bar{D}_{\dot{1}t} X^{(2)} \tag{30}$$

,

and similarly, (16',17') may be solved in terms of two further superfields $Y^{(1)}$ and $Y^{(2)}$ defined by

$$B^{-1} D_1^t B = D_2^t Y^{(1)} \tag{31}$$

$$B^{-1} \bar{D}_{\dot{1}t} B = \bar{D}_{\dot{2}t} Y^{(2)} \tag{32}$$

.

We now define functions U_α^s, V_α^s, $U_{\dot{\alpha}t}$, $V_{\dot{\alpha}t}$ by the relations:

$$\begin{aligned}
V_\alpha^1 &= U_\alpha^3 \equiv D_\alpha^2 Z^{(1)} \\
V_\alpha^2 &= V_\alpha^3 \equiv D_\alpha^1 Z^{(1)} \\
U_\alpha^1 &= U_\alpha^2 \equiv D_\alpha^3 Z^{(1)} \\
\\
V_{\dot{\alpha}1} &= U_{\dot{\alpha}3} \equiv \bar{D}_{\dot{\alpha}2} Z^{(2)} \\
V_{\dot{\alpha}2} &= V_{\dot{\alpha}3} \equiv \bar{D}_{\dot{\alpha}1} Z^{(2)} \\
U_{\dot{\alpha}1} &= U_{\dot{\alpha}2} \equiv \bar{D}_{\dot{\alpha}3} Z^{(2)}
\end{aligned} \tag{33}$$

where for $D_A^s = (D_\alpha^s , \bar{D}_{\dot{\alpha}t})$,

$$\begin{aligned}
D_A^s Z^{(n)} &= D_A^s X^{(n)} \quad \text{for } A = 1, \dot{1} \\
&= D_A^s Y^{(n)} \quad \text{for } A = 2, \dot{2}
\end{aligned}$$

.

The first continuity equation immediately follows since $j = U + V$ is a conserved spinor current satisfying

$$\sum_s \left(D_1^s j_1^s + D_2^s j_2^s + \bar{D}_{\dot{1}s} j_{\dot{1}s} + \bar{D}_{\dot{2}s} j_{\dot{2}s} \right) = 0 \quad . \tag{34}$$

It is now clear that an infinity of currents $j^{(n)}$, $n = 1, \ldots\ldots\ldots, \infty$, may be iteratively constructed using the functions $U_\alpha^{(n)s}$, $U_{\dot{\beta}t}^{(n)}$, $V_\alpha^{(n)s}$, $V_{\dot{\beta}t}^{(n)}$ defined by relations of the form (33) with superfields $X^{(n)}$, $Y^{(n)}$ satisfying the recursion relations :

$$D_1^s X^{(n)} = \left[D_2^s + (B D_2^s B^{-1}) \right] X^{(n-1)} \tag{35}$$

$$\bar{D}_{\dot{1}s} X^{(n)} = \left[\bar{D}_{\dot{2}s} + (B \bar{D}_{\dot{2}s} B^{-1}) \right] X^{(n-2)} \tag{36}$$

$$D_2^s Y^{(n)} = \left[D_1^s + (B^{-1} D_1^s B)\right] Y^{(n-1)} \tag{37}$$
$$\overline{D}_{2s} Y^{(n)} = \left[D_{is} + (B^{-1}\overline{D}_{is} B)\right] Y^{(n-2)} \tag{38}$$

We now denote $\mathcal{J}$, U, V by $\mathcal{J}^{(1)}$, $U^{(1)}$, $V^{(1)}$ respectively. Setting $X^{(0)} = \mathbb{1} = Y^{(0)}$ in (35-38) yields the previous relations (29-32); and it is easy to show following the above procedure and using the equations of motion (16'-19', 16-19), that relations (35-38) provide a continuity equation of the form of (34) for each n.

The recursion relations (35-38) may be obtained from (20,21,20',21') by performing the power series expansions

$$\Psi = \sum_{n=0}^{\infty} \lambda^n X^{(n)} \tag{39}$$
$$\Phi = \sum_{n=0}^{\infty} \lambda^{-n} Y^{(n)} \tag{40}$$

It is therefore clear that a complete set of functions $X^{(n)}, Y^{(n)}$ $(n = 1,\ldots,\infty)$ will exist as long as Ψ is analytic around the origin of the complex plane, and Φ is analytic in a region containing $\lambda = \infty$. We note that in addition to (36,38), the linear equations (21,21'), with Ψ and Φ given by (39,40) also imply that the original functions $X^{(1)}$ and $Y^{(1)}$ satisfy:

$$\overline{D}_{is} X^{(1)} = 0 = \overline{D}_{2s} Y^{(1)} \tag{41}$$

Further, the relations (29-32) solve all the equations (16-19,16'-19') as long as Ψ and Φ, represented by the expansions (39,40), satisfy (22,22'). We explicitly check this claim for the case of eq.(18) by inserting (30) :

$$\begin{aligned} D_1^s (B D_{2t} B^{-1}) &= - D_1^s D_{it} X^{(2)} \\ &= \delta_t^s \, 2i \, \partial_{1i} X^{(2)} + D_{it} D_1^s X^{(2)} \\ &= \delta_t^s \, 2i \left(\partial_{1i} X^{(2)} + \partial_{2i} X^{(1)}\right) + D_{it} (B D_2^s B^{-1}) X^{(1)}, \end{aligned}$$

where we have used (35) and (41); and using (29) we obtain

$$\begin{aligned} D_1^s (B D_{2t} B^{-1}) &= \delta_t^s \, 2i \left(\partial_{1i} X^{(2)} + \partial_{2i} X^{(1)} - \partial_{1i} X^{(1)} . X^{(1)} \right) \\ &= \delta_t^s \, 2i \left(\partial_{1i} X^{(2)} + \partial_{2i} X^{(1)} + g \nabla_{2i} g^{-1} . X^{(1)} \right), \end{aligned}$$

where use has been made of the coefficient of λ^{-1} in the expansion of (22). Now using the λ-independent piece of (22), viz.,

$$\partial_{1i} X^{(2)} + (\partial_{2i} + g \nabla_{2i} g^{-1}) X^{(1)} + g \nabla_{1i} g^{-1} = 0 ,$$

we obtain

$$D_1^s\left(B\, D_{\dot{2}t}\, B^{-1}\right) \;=\; -2i\,\delta_t^s\, g\,\nabla_{1\dot{2}}\, g^{-1} \qquad \text{, i.e. eq.(18),}$$

verifying that the expressions (29,30) solve (18).

We have seen that the expressions (29-32) which solve the constraint equations are just the λ-independent parts of the linear equations (20,21,20',21'). We may generalize (29-32) by writing a λ-dependent solution of (16-19,16'-19') :

$$\left(B D_2^s B^{-1}\right) \;=\; \lambda^{-1}\,\Psi\left(D_1^s + \lambda D_2^s\right)\Psi^{-1} \tag{42}$$

$$\left(B \bar{D}_{\dot{2}t} B^{-1}\right) \;=\; \Psi\left(\bar{D}_{\dot{2}t} + \lambda^{-2}\bar{D}_{\dot{1}t}\right)\Psi^{-1} \tag{43}$$

$$\left(B^{-1} D_1^s B\right) \;=\; \lambda\,\Phi\left(D_2^s + \lambda^{-1} D_1^s\right)\Phi^{-1} \tag{44}$$

$$\left(B^{-1}\bar{D}_{\dot{1}t} B\right) \;=\; \Phi\left(\lambda^2\bar{D}_{\dot{2}t} + \bar{D}_{\dot{1}t}\right)\Phi^{-1} \;, \tag{45}$$

where Ψ, Φ satisfy (20-22,20'-22').

Transformations from one order in λ to another on the right of (42-45) correspond to Backlund transformations since they relate two solutions of the equations (16-19,16'-19'). The infinitesimal forms of these transformations are the symmetry transformations responsible for the above infinite set of nonlocal conserved currents. The situation here, as in all the cases we've considered in previous chapters, is very reminiscent of a common feature of all the two-dimensional integrable soliton theories: the transformations which generate the conservation laws provide the key to the transformations which generate exact solutions. Under these (infinitely many) infinitesimal transformations we may define the variation of the gauge-invariant superfield B to be of the form

$$\delta^{(n)}B \;=\; -\left(S^{(n)}B + BR^{(n)}\right) \quad ,\ n\in\mathbb{Z},\quad \begin{array}{l} S^{(n)} = 0 \ \text{ for } n<0, \\ R^{(n)} = 0 \ \text{ for } n>0; \end{array} \tag{46}$$

where $S^{(n)}, R^{(n)}$ are Lie algebra valued nonlocal functions.

In terms of the generating functions for $S^{(n)}$ and $R^{(n)}$:

$$S \;=\; \sum_{n=0}^{\infty}\lambda^n S^{(n)} \quad , \qquad R \;=\; \sum_{n=0}^{\infty}\lambda^{-n} R^{(n)} \quad ,$$

we may write (46) as

$$\delta B \;=\; \sum_{n=-\infty}^{\infty}\delta^{(n)} B \;=\; -\left(SB + BR\right) \,. \tag{47}$$

We shall now show that these transformations are symmetries of equations (16-19, 16'-19') if S and R have the familiar structure:

$$S = \Psi(\lambda)\, T\, \Psi(\lambda)^{-1}$$
$$R = \Phi(1/\lambda)\, T\, \Phi(1/\lambda)^{-1} \tag{48}$$

where Ψ and Φ satisfy (20-22) and (20'-22') respectively; and T is a constant Lie algebra valued matrix. We first consider transformations (46) for $n \geqslant 0$. The generating function S given by (48a) for these infinitesimal transformations satisfies the equations

$$(D_1^s + \lambda D_2^s) S = -\lambda \left[B D_2^s B^{-1}, S \right] \tag{49a}$$

$$(\bar{D}_{\dot{2}t} + \lambda^{-2} \bar{D}_{\dot{1}t}) S = - \left[B \bar{D}_{\dot{2}t} B^{-1}, S \right] \tag{49b}$$

$$(\partial_{1\dot{2}} + \lambda\, \partial_{2\dot{2}} + \lambda^{-1} \partial_{2\dot{1}} + \lambda^{-2} \partial_{1\dot{1}}) S = -\left[(g \nabla_{1\dot{2}} g^{-1} + \lambda B \partial_{2\dot{2}} B^{-1} + \lambda^{-1} g \nabla_{2\dot{1}} g^{-1}), S \right] \tag{49c}$$

by virtue of eqs(20-22). Now, under the transformation $\delta B = -SB$, the variation of eq.(16) is given by

$$D_1^{(s} \delta (B D_2^{t)} B^{-1}) = D_1^{(s} D_2^{t)} S + D_1^{(s} \left[B D_2^{t)} B^{-1}, S \right]$$
$$= - D_2^{(t} D_1^{s)} S - \left\{ B D_2^{(t} B^{-1}, D_1^{s)} S \right\} \quad \text{, using (16);}$$
$$= 0 \quad \text{, as a result of the consistency of eqs.(49a).}$$

Similarly, (17) is invariant under these transformations as a consequence of the consistency of (49b); and the invariance of (18, 19) follows from (49a-c). For instance, we consider the variation of (18):

$$D_1^s \, \delta (B \bar{D}_{\dot{2}t} B^{-1}) = D_1^s \bar{D}_{\dot{2}t} S + D_1^s \left[B \bar{D}_{\dot{2}t} B^{-1}, S \right]$$
$$= -2i\, \delta^s{}_t \left(\partial_{1\dot{2}} S + [g \nabla_{1\dot{2}} g^{-1}, S] \right)$$
$$- \left(\bar{D}_{\dot{2}t} D_1^s S + \left\{ B \bar{D}_{\dot{2}t} B^{-1}, D_1^s S \right\} \right) \quad \text{, using (18);}$$
$$= -2i\, \delta^s_t \left(\partial_{1\dot{2}} S + [g \nabla_{1\dot{2}} g^{-1}, S] \right) , \tag{50}$$

since the remaining terms vanish as a consequence of the consistency of (49). Now, since the transformation $\delta B = -SB$ is effected by the infinitesimal transformations: $\delta g = -Sg$, $\delta h = 0$; we note that

$$\delta (g \nabla_{1\dot{2}} g^{-1}) = \partial_{1\dot{2}} S + \left[g \nabla_{1\dot{2}} g^{-1}, S \right] .$$

Eq.(50) therefore implies the invariance of (18).

We may now consider transformations (46) for $n \leqslant 0$ given by $\delta B = -BR$, resulting from the tranformations $\delta g = 0$, $\delta h = Rh$; with R given by (48) satisfying

$$[L'^s, S] = 0 = [M'_t, S] = [N', S] \tag{51}$$

as a consequence of (20'–22'). It is clear that these infinitesimal transformations will leave (16'–19') invariant as a consequence of (51).

Above, we have used the fact that under the transformation (46),

$$\begin{aligned}\delta(BD_2^s B^{-1}) &= D_2^s S(\lambda) + [BD_2^s B^{-1}, S(\lambda)] + BD_2^s R(\lambda) B^{-1} \\ &= -\tfrac{1}{\lambda} D_1^s S(\lambda) + BD_2^s R(\lambda) B^{-1} \ .\end{aligned} \tag{52}$$

Using (52) we may now evaluate the variation due to (46) of the functionals Ψ and Φ, which are formally path-ordered exponential functions of, for instance, $BD_2^s B^{-1}$ and $B^{-1}D_1^s B$, from (20) and (20'), respectively. We choose to split (46), denoting transformations corresponding to $n \geqslant 0$ by

$\delta^+(\lambda) B = -S(\lambda) B$; and those corresponding to $n \leqslant 0$ by

$\delta^-(\lambda) B = -B R(\lambda)$. The change in Ψ and Φ may now be determined iteratively (following the procedure of [33,34]) starting from

$$\delta^+(\mu) X^{(1)} = -\mu^{-1} S(\mu) \quad , \quad \delta^-(\mu) Y^{(1)} = \mu R(\mu) \ ,$$

which may be obtained directly from (29,31) (the first terms in the power series expansions of (20,20')). Explicitly,

$$D_1^s \delta^+ X^{(1)} = \delta^+(BD_2^s B^{-1}) = -\mu^{-1} D_1^s S(\mu) \qquad \text{, from (52);}$$

and similarly,

$$D_2^s \delta^- Y^{(1)} = \delta^-(B^{-1} D_1^s B) = \mu D_2^s R(\mu)$$

We may similarly evaluate the change in every coefficient $X^{(n)}$, $Y^{(n)}$ using the recursion relations (35,37); and summing these variations we may obtain

$$\begin{aligned}\delta^+(\mu)\Psi(\lambda) &= \sum_{n=0}^{\infty} \lambda^n \delta(\mu) X^{(n)} \equiv \Psi[\lambda; B+\delta^+(\mu)B] - \Psi[\lambda; B] \\ &= \frac{\lambda}{\mu-\lambda}\big(S(\mu) - S(\lambda)\big)\Psi(\lambda)\end{aligned} \tag{53a}$$

$$\delta^-(\mu)\Phi(\lambda) = \frac{-\lambda}{\mu-\lambda}\big(R(\mu) - R(\lambda)\big)\Phi(\lambda) \tag{53b}$$

$$\delta^-(\mu)\Psi(\lambda) = \frac{\lambda}{\mu-\lambda}\big(B^{-1}R(\mu)B - S(\lambda)\big)\Psi(\lambda) \tag{53c}$$

$$\delta^+(\mu)\Phi(\lambda) = \frac{-\lambda}{\mu-\lambda}\big(B\,S(\mu)B^{-1} - R(\lambda)\big)\Phi(\lambda) \ . \tag{53d}$$

The proof of these expressions follows that for the analogous expressions in the models of chapters 2 & 3. We now observe that because of the form of the variations (53), the infinitesimal transformation (47) is a symmetry of the linear system (20-22, 20'-22'); and this symmetry of the linear system is the source of the hidden symmetry of the constraint equations (16-19, 16'-19'). We explicitly demonstrate the invariance of the linear equations in two specific cases; the invariance of the other equations being verifiable in a similar fashion. We first consider the first-order change in (20) due to the transformation $\delta^+(\mu)B = -S(\mu)B$:

$$\begin{aligned}\delta^+(\mu)\left\{L^s(\lambda)\Psi(\lambda)\right\} &= (D_1^s+\lambda D_2^s)\,\delta^+(\mu)\Psi(\lambda) + \lambda\,\delta^+(\mu)(BD_2^sB^{-1}).\,\Psi(\lambda)\\ &\quad + \lambda(BD_2^sB^{-1})\,\delta^+(\mu)\Psi(\lambda)\\ &= \left\{\frac{\lambda}{\mu-\lambda}(D_1^s+\lambda D_2^s)S(\mu) - \frac{\lambda}{\mu}D_1^sS(\mu) + \frac{\lambda^2}{\mu-\lambda}\left[BD_2^sB^{-1},\,S(\mu)\right]\right\}\Psi(\lambda),\end{aligned}$$

using (49a), (53a) and (20);

$$= \frac{\lambda^2}{\mu(\mu-\lambda)}\left\{(D_1^s+\mu D_2^s)S(\mu) + \mu\left[BD_2^sB^{-1},\,S(\mu)\right]\right\}$$

$= \quad 0$, by (49a).

Similarly,

$$\begin{aligned}\delta^-(\mu)\left\{L^s(\lambda)\Psi(\lambda)\right\} &= (D_1^s+\lambda D_2^s)\,\delta^-(\mu)\Psi(\lambda) + \lambda\,\delta^-(\mu)(BD_2^sB^{-1})\,\Psi(\lambda)\\ &\quad + \lambda(BD_2^sB^{-1})\,\delta^-(\mu)\Psi(\lambda)\\ &= \left\{\frac{\lambda}{\mu-\lambda}(D_1^s+\lambda D_2^s)(B^{-1}R(\mu)B - S(\lambda)) - \frac{\lambda^2}{\mu-\lambda}\left[(B^{-1}R(\mu)B - S(\lambda)),\,BD_2^sB^{-1}\right]\right.\\ &\qquad \left. + \lambda\,BD_2^sR(\mu)B^{-1}\right\}\Psi(\lambda),\end{aligned}$$

from (53c), (52) and (20);

$$\begin{aligned}= \frac{\lambda}{\mu-\lambda}\Big\{ & B^{-1}(D_1^s+\lambda D_2^s)R(\mu)B + \lambda\,BD_2^sRB^{-1}\\ & - \left[(B^{-1}D_1^sB + \lambda B^{-1}D_2^sB),\,B^{-1}RB\right]\\ & - \lambda\left[B^{-1}R(\mu)B,\,BD_2^sB^{-1}\right]\Big\}\Psi\end{aligned}$$

, using (49b);

$$= \frac{\lambda}{\mu-\lambda}\left\{B^{-1}(D_1^s+\mu D_2^s)R(\mu)B + B^{-1}\left[B^{-1}D_1^sB,\,R(\mu)\right]B\right\}$$

$= \quad 0$, by the expression for R analogous to (49a).

The structure of the symmetry transformations displayed by (53) is almost identical to that discussed in chapter 3 for the case of the self-duality equations [55] . It is therefore clear that we may define the infinitesimal generators of these symmetries:

$$M_a = -\int d^4x\, d^{2N}\theta\, d^{2N}\bar{\theta} \quad \delta_a B \frac{\delta}{\delta B}$$
$$= \int d^4x\, d^{2N}\theta\, d^{2N}\bar{\theta} \quad (S_a B + B R_a)\frac{\delta}{\delta B} \qquad ,$$

where we have expanded the transformations in a basis of the Lie algebra G; and considering the composition of two such symmetry operations (clearly given by the Lie bracket):

$$[M_a(\lambda), M_b(\mu)] = \mathcal{L}_{M_a(\lambda)} M_b(\mu)$$
$$= \int d^4x d^{2N}\theta d^{2N}\bar{\theta}\ \left[M_a(\lambda), (S_b(\mu)B + BR_b(\mu))\frac{\delta}{\delta B}\right] \quad (54)$$

we may, using (53) and following the arguments of chapters 2, 3.1 [45,55], show that the coefficients of $\lambda^m\mu^n$ on both sides of (54) realize the loop algebra $G\otimes\mathbb{C}[\lambda,\lambda^{-1}]$, (whose elements are Laurent polynomials in λ with coefficients in the simple lie algebra G), with commutation relations

$$[M_a^m, M_b^n] = M_c^{m+n}\, c_{abc} \quad , \quad -\infty < m,n < \infty \quad , \qquad (55)$$

where c_{abc} are the structure constants of G;and the M_a^n are coefficients in the Laurent expansion of M_a, the infinitesimal operators of the symmetry group:

$$M_a = \sum_{n=-\infty}^{\infty} \lambda^n M_a^n \quad .$$

The remarkable similarity to the self-dual pure gauge theory displayed by the features described in this chapter suggests that a solution on the lines of [17] is possible. With this tantalizing prospect in mind, we note that the matrix

$$G \equiv \Psi^{-1} B \Phi = (\Psi^{-1}g)(h^{-1}\Phi) = \psi^{-1}\varphi \qquad (56)$$

is an integrable phase factor; and identically satisfies

$$\begin{aligned} (D_1^S + \lambda D_2^S)G &= 0 \\ (\bar{D}_{2t} + \lambda^{-1}\bar{D}_{1t})G &= 0 \\ (\partial_{1\dot{2}} + \lambda\partial_{2\dot{2}} + \lambda^{-1}\partial_{2\dot{1}} + \lambda^{-2}\partial_{1\dot{1}})G &= 0 \end{aligned} \qquad (57)$$

as a consequence of eqs(20-22,20'-22'). However, the geometrical significance of (57) is not clear. In particular, whether or not gauge potentials constructed from the factors of G in (56) would correspond to vector bundles over some supersymmetric twistor space [81] is not immediately apparent. However, in a previous paper [74], Volovich has suggested a supersymmetric version of the ADHM construction [18], where the spinor connection, in addition to (5), satisfies the equation:

$$F_{\mu,\beta t} = 0 \quad , \tag{58}$$

which, taken together with (5), implies that the vector superfield curvature is automatically self-dual: $F_{\mu\nu} = {}^{*}F_{\mu\nu}$. In [73] Volovich also indicated how the linear system (24-26) could be modified so as to incorporate (58). In our formulation, this modified system may be written:

$$(D_1^s + \lambda D_2^s + \lambda B D_2^s B^{-1})\Psi = 0 = (\bar{D}_{\dot{2}t} + B\bar{D}_{\dot{2}t}B^{-1})\Psi \quad , \quad \bar{D}_{\dot{1}t}\Psi = 0 \quad ,$$

$$[\partial_{1\dot{2}} + g\nabla_{1\dot{2}}g^{-1} + \lambda(\partial_{2\dot{2}} + g\nabla_{2\dot{2}}g^{-1})]\Psi = 0 \tag{59}$$

$$[\partial_{1i} + \lambda(\partial_{2i} + g\nabla_{2i}g^{-1})]\Psi = 0 \quad . \tag{60}$$

We note that (22) has been split into two equations (59,60). Now, compatibility of all these equations clearly requires that

$$A_{1\dot{2}} = g^{-1}\partial_{1\dot{2}}g \quad , \quad g\nabla_{1\dot{2}}g^{-1} = 0 \quad , \quad A_{2i} = h^{-1}\partial_{2i}h \quad ;$$

yielding

$$[\partial_{1\dot{2}} + \lambda(\partial_{2\dot{2}} + B\partial_{2\dot{2}}B^{-1})]\Psi = 0 \tag{59'}$$

$$[\partial_{1i} + \lambda(\partial_{2i} + B\partial_{2i}B^{-1})]\Psi = 0 \quad ; \tag{60'}$$

a form which clearly implies that the superfield $F_{\mu\nu}$ is self-dual;(c.f. section 3.1). Using (59',60') we may generate, following the procedure of section 3.1, an infinity of vector supercurrents J_μ(satisfying $\partial_\mu J^\mu = 0$), in addition to the spinorial currents which may be deduced from those displayed above. All these form a supermultiplet satisfying the conservation law: $D_\alpha^s J_s^\alpha + \bar{D}_{\dot{\alpha}t} J^{\dot{\alpha}t} + \partial_\mu J^\mu = 0$.

To conclude, we recall that the constraint equations (5) only imply the Yang-Mills equations for the maximally extended (N = 3,4) case. For this case, therefore, the conservation laws we have displayed are analogous to the conserved currents to be found in two dimensional completely integrable supersymmetric models [75-79], and we may expect them to be of relevance for the quantum theory; particularly for the finiteness of the maximally extended theory. However, the implications of the features displayed in this chapter for the N = 1 and N = 2 theories remain obscure.

Chapter 5: Gauge theories on a straight-line path.

We consider the non-integrable phase factor of gauge theories [12] on a line connecting points x and y in (complexified) euclidean space:

$$\begin{aligned}\Psi_{x,y} &= P\, e^{\int_x^y A\cdot dx} = \lim_{N\to\infty} \Psi_{x,x_1}\Psi_{x_1,x_2}\Psi_{x_2,x_3}\cdots\cdots\Psi_{x_{N-1},x_N}\Psi_{x_N,y} \\ &= \lim_{N\to\infty} e^{A(x)\cdot(x_1-x)}\, e^{A(x_1)\cdot(x_2-x_1)}\cdots\cdots\cdots\, e^{A(x_N)\cdot(y-x_N)} \\ &= \lim_{N\to\infty}\left\{(1+A(x)\cdot(x_1-x))(1+A(x_1)\cdot(x_2-x_1))\cdots\cdots(1+A(x_N)\cdot(y-x_N))\right\} \end{aligned} \tag{1}$$

with a view to studying its path dependence (i.e. non-integrability). We recall that for (anti-)self-dual fields, with the gauge connection taking the form [18]

$$A_\mu(x) = v(x)^\dagger \partial_\mu v(x) \quad , \quad v^\dagger v = 1 \quad , \tag{2}$$

the path-ordered phase-factor has the manifestly path-independent representation [86]

$$\Psi_{x,y} = v^\dagger(x)\, v(y) \quad , \tag{3}$$

where the path between x and y is restricted to lie on a null-plane in complexified space. This is just a consequence of the fact [15] that the self-duality equations are just a statement of the vanishing of the gauge curvature on anti-dual null planes in $\mathbb{C}^4$. The form (2) for local gauge potentials is valid in the general case (i.e. it is not specific to self-dual connections) (see [87] and references therein), and we shall use it in what follows. Here $v(x)$ is a complex $N\times p$ matrix satisfying $v^\dagger v = \mathbb{1}_p$; and a right action on v by an element of the gauge group corresponds to a local gauge transformation of (2). We shall also use the manifestly gauge-invariant $N\times N$, rank p projection operator $P(x) = v\, v^\dagger$, which projects onto the N-dimensional subspace of $\mathbb{C}^{N+p}$ spanned by the column vectors of v. $P(x) = P(x)^\dagger$.

We begin by considering (1) with $y = x + 2a$, where a is an infinitesimally small distance. Then,

$$v(x)\,\Psi_{x,x+2a}\, v(x+2a)^\dagger = v(x)\, e^{A(x)\cdot(2a)}\, v(x+2a)^\dagger .$$

We write $A(x) = v^\dagger v'$, $v \equiv v(x)$, $v_{2a} \equiv v(x+2a)$, where the prime denotes differentiation along the line from x to x+2a.

Then, performing a Taylor expansion about x, and expanding the exponential in a power series in a, we obtain, to $O(a^2)$:

$$
\begin{aligned}
V\,\Psi_{x,x+2a}\,V^{\dagger}_{2a} &\simeq v\, e^{2a v^{\dagger} v'}\left(v^{\dagger} + 2a v^{\dagger\prime} + 2a^2 v^{\dagger\prime\prime}\right) \\
&= v\left(1 + 2a v^{\dagger}v' + 2a^2 v^{\dagger}v'v^{\dagger}v'\right)\left(v^{\dagger} + 2a v^{\dagger\prime} + 2a^2 v^{\dagger\prime\prime}\right) \\
&= P + 2a\,PP' + 2a^2 PP'' - a^2 PP''P \\
&\quad + a^2\left(v v^{\dagger\prime\prime} P - P v'' v^{\dagger}\right)
\end{aligned}
\tag{4}
$$

where $P \equiv P(x) = v\,v^{\dagger}$, and $P^2 = P$ implies $P\,P'P = 0$.

We note that since (4) is hermitian, we have

$$v v^{\dagger\prime\prime} P = P v'' v^{\dagger} \quad ; \quad P^{\dagger} = P \; .$$

Also, since $v^{\dagger}P = v^{\dagger}$, we may, at will, multiply (4) on the right by $P_{2a} \equiv P(x+2a)$.

We now observe that

$$
\begin{aligned}
P(x)\left(2P(x+a)-1\right)P(x+2a) &\equiv P\left(2P_a - 1\right)P_{2a} \\
&\simeq P\left(2P - 1 + 2aP' + a^2 P''\right)\left(P + 2aP' + 2a^2 P''\right)P_{2a} \\
&= P\left(P + 2aPP' + a^2 PP''\right)\left(P + 2aP' + 2a^2 P''\right)P_{2a} \\
&\simeq P\left(P + 2aPP' + 2a^2PP'' + 4a^2 PP'P + a^2 PP''P\right)P_{2a} \\
&= P\left(P + 2aPP' + 2a^2PP'' - a^2 PP''P\right)P_{2a}, \quad \text{since } P\,P'P = 0.
\end{aligned}
$$

Comparing the latter relation with (4), we make the identification:

$$
\begin{aligned}
V(x)\,\Psi_{x,x+2a}\,V^{\dagger}(x+2a) &\simeq P(x)\left(2P(x+a)-1\right)P(x+2a) \\
&= -P(x)\exp\left(i\pi P(x+a)\right)P(x+2a) .
\end{aligned}
\tag{5}
$$

Now, to order a^2, we note several equivalent forms of the right-hand side of (5) :

$$
\begin{aligned}
P(x)\left(2P(x+a)-1\right)P(x+2a) &= P(x)\left(\tfrac{1}{2} - \tfrac{1}{2}P(x+2a)P(x) + P(x)P(x+2a)\right)P(x+2a) \\
&= \left(\tfrac{3}{2}P(x)P(x+2a) - \tfrac{1}{2}P(x)P(x+2a)P(x)P(x+2a)\right) \\
&= P(x)\left(1 + \tfrac{1}{2}\left[P(x), P(x+2a)\right]\right)P(x+2a) \; ,
\end{aligned}
$$

which may be checked by explicitly expanding about x up to $O(a^2)$.

We remark that the form of (5) is consistent with the representation of the phase factor in terms of the projection operators P [88] :

$$V(x)\,\Psi_{x,y}\,V^{\dagger}(y) \;=\; P(x)\,e^{\int_x^y [P,\,\partial_\mu P]\,dx^\mu}\,P(y) \;, \qquad (6)$$

since

$$P(x)\,e^{2a[P(x),\,P'(x)]}\,P(x+2a)$$
$$\simeq P(x)\Big(1 + 2a\,[P(x), P'(x)] + 2a^2\big([P(x), P'(x)]\big)^2\Big)P(x+2a)$$
$$= P(x)\big(P(x) + 2a\,P(x)P'(x) - 2a^2\,P'(x)P'(x)\big)\big(P(x) + 2a\,P'(x) + 2a^2 P''(x)\big)\,P_{(x+2a)}$$
$$= P(x)\,\big(2P(x+a) - 1\big)\,P(x+2a) \;.$$

We note that in the self-dual sector, when the path is restricted to lie on a null-plane, we have [86] :

$$P(x)\,P(y)\,P(z) \;=\; P(x)\,P(z) \qquad , \text{ for any } x, y, z \text{ on the path.}$$

Equation (5) therefore reduces to the known form (3) for the case of self-dual fields. However, there do not seem to be any other (non-trivial) situations in which the phase-factor takes the form

$$\Psi_{x,y} \;=\; F_{x,y} \;=\; \big(F(C)^{-1}\big)_{x,-\infty}\;\big(F(C')\big)_{-\infty,y} \qquad (7)$$

where the two paths C , C' only necessarily coincide between x and y.
If such a factorizable $F_{x,y}$ could be found, then the potential

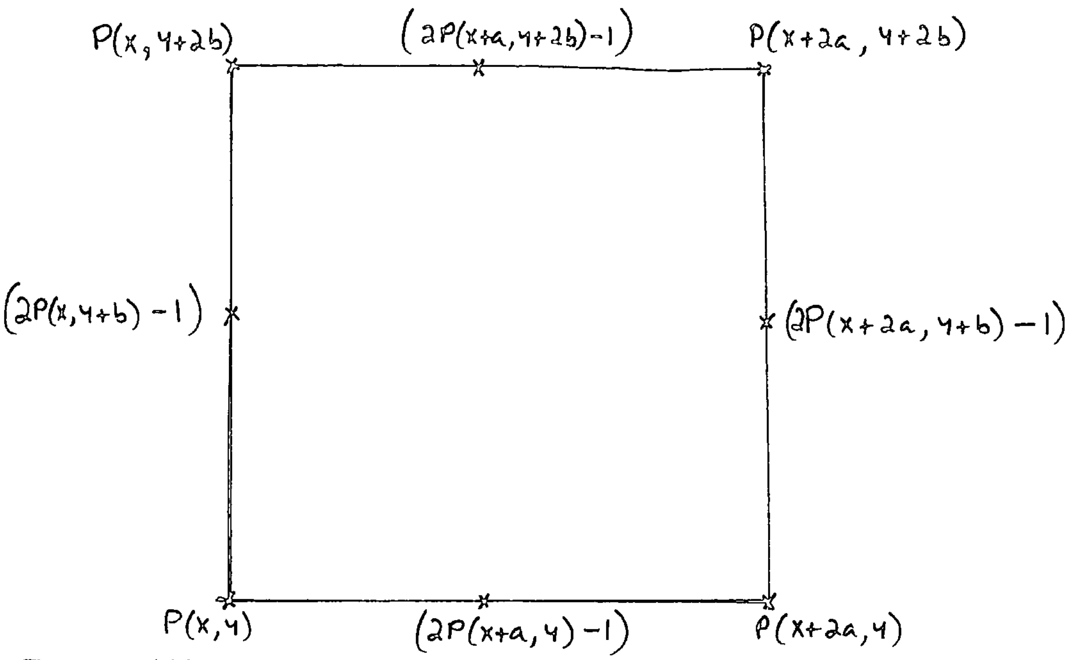
$$A \;=\; F^{-1}\,\partial_\mu\,F$$

would clearly be pure-gauge since the phase factors along the two paths C , C' from $-\infty$ to x in (7) would have to cancel. We would then have an integrable sector of the theory. In this context, it would be interesting to consider the supersymmetric generalization of this formulation in view of Witten's discussion [72] of the constraint equations of the N = 3 theory in terms of integrability on lines.

We remark that in the general case, the full Yang-Mills equations have been numerically shown to be non-integrable by Nikolaevski and Shur [90] . They have considered a particular one dimensional reduction of the SU(2) theory and have shown that the equations of motion have no integrals of motion apart from the hamiltonian.

We shall now show that the approximation for the phase factor given by (5) yields the correct continuum action to $O(a^4)$ when inserted into Wilson's formula [68] . After this work was completed, we noticed that

Fröhlich [89] had also attempted to write down a lattice action in terms of these projection operators. However, the action he suggests does not have the correct continuum limit [91]). We consider the trace of the product of all the objects around the elementary plaquette:



Then, writing

$$P_1 \equiv a\,\partial_x P \quad , \quad P_{11} \equiv a^2 \partial_x \partial_x P \quad , \quad P_2 \equiv b\,\partial_y P ,$$
$$P_{22} \equiv b^2 \partial_y \partial_y P \quad , \quad P_{12} \equiv ab\, \partial_x \partial_y P \; ; \; (a = b),$$

and using the identities implied by $P^2 = P$;

e.g.

$$P P_a = P_a (1 - P) \quad , \quad a = 1, 2 .$$
$$P P_a P = 0$$
$$P P_{aa} P = -2 P P_a P_a P \quad (\text{no sum over } a)$$
$$= -2 P P_a P_a$$

$$P P_{1122} P + 2 P P_2 P_{112} + 2 P P_1 P_{122} + 2 P P_{112} P_2 + 2 P P_{122} P_1$$
$$= -\left(4 P P_{12} P_{12} + P P_{11} P_{22} + P P_{22} P_{11}\right) \qquad (8)$$

and also using hermiticity and the cyclic property of the trace, which yield for instance from (8), the relation

$$\mathrm{tr}\left(P P_{1122} + 4 P P_{122} P_1 + 4 P P_{112} P_2\right) = -\mathrm{tr}\left(4 P P_{12} P_{12} + 2 P P_{11} P_{22}\right),$$

it is staightforward, though tedious, to verify that

$$
\begin{aligned}
&\mathrm{tr}\ \Big\{ P(x,y)\big(2P(x+a,y)-1\big)P(x+2a,y)\big(2P(x+2a,y+b)-1\big)\cdot \\
&\quad \cdot P(x+2a,y+2b)\big(2P(x+a,y+2b)-1\big)P(x,y+2b)\cdot \\
&\quad \cdot\big(2P(x,y+b)-1\big)P(x,y)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\simeq \mathrm{tr}\ \Big\{ & P(2P+2P_1+P_{11}-1)(P+2P_1+2P_{11})\cdot \\
&\cdot(2P+4P_1+2P_2+4P_{11}+P_{22}+4P_{12}+4P_{112}+2P_{122}+2P_{1122}-1)\cdot \\
&\cdot(P+2P_1+2P_2+2P_{11}+2P_{22}+4P_{12}+4P_{112}+4P_{122}+4P_{1122})\cdot \\
&\cdot(2P+4P_2+2P_1+4P_{22}+P_{11}+4P_{12}+4P_{122}+2P_{112}+2P_{1122}-1)\cdot \\
&\cdot(P+2P_2+2P_{22})(2P+2P_2+P_{22}-1)\Big\}; \qquad (P\equiv P(x,y))
\end{aligned}
$$

when expanded to order a^2b^2, yields terms proportional to:

$$
\begin{aligned}
&\mathrm{tr}\ (2P_1P_2P_1P_2P - 2P_2P_1P_1P_2P) \\
&= \mathrm{tr}\ (P_1P_2P_1P_2P + P_2P_1P_2P_1P - P_2P_1P_1P_2P - P_1P_2P_2P_1P),
\end{aligned}
$$

$$
= \mathrm{tr}\quad P[P_1,P_2]P[P_1,P_2]P
$$

i.e. the continuum action; (since $F_{\mu\nu} = V^{\dagger}[P_\mu,P_\nu]V$).

Chapter 6 : Gauge theories in dimensions greater than four.

In this chapter we obtain equations for euclidean gauge theories in higher dimensions which are analogues of the self-duality equations in the sense that they are linear algebraic relations amongst the components of the field strength tensor which put the pure gauge theory on-shell (i.e. they imply the source-free Yang-Mills equations as a consequence of the Bianchi identity).

(i) We first recall some facts about the self-duality equations in order to obtain some clues as to their possible generalization. We note that since the Levi-Civita tensor $\epsilon_{\mu\nu\rho\sigma}$ is SO(4) invariant, the self-duality relations have the maximal space-time symmetry. Now, SO(4) is locally equivalent to SO(3)⊗SO(3), and the antisymmetric tensors in SO(4) (having 6 components) form a $3 + \bar{3}$ representation of SO(3)⊗SO(3) ; the (anti-) self-dual tensors transforming as a 3-vector of one of the SO(3) groups. One may therefore consider three self-dual generators of SO(4) $\eta^{(+)a}_{\mu\nu}$ (a = 1,2,3; = 1,....,4) which generate one SO(3) and three anti-dual ones $\eta^{(-)a}_{\mu\nu}$ ($\eta^{(\pm)a}_{\mu\nu} = \pm\epsilon_{\mu\nu\rho\sigma}\,\eta^{(\pm)a}_{\rho\sigma}$) which generate the other SO(3), defined by [93] :

$$\eta^{(\mp)a}_{\mu\nu} = \pm\,\epsilon_{4a\mu\nu} + \delta_{a\mu}\delta_{\nu 4} - \delta_{a\nu}\delta_{\mu 4} \,, \tag{1}$$

as tensors which map antisymmetric representations of SO(4) onto vectors of one of its two invariant SO(3) subgroups. The tensors (1), regarded as 4×4 matrices realize the quaternion algebra

$$\eta^a\,\eta^b = -\delta_{ab}\,\mathbb{1} + \epsilon_{abc}\,\eta^c \,. \tag{2}$$

The definition (1) implies that the (anti) self-duality equations may be written

$$\eta^{(\pm)a}_{\mu\nu}\,F_{\mu\nu} = 0 \,. \tag{3}$$

We note that $\eta^{(\pm)a}$ form complete sets of real mutually anticommuting antisymmetric matrices with square -1 . From (1) we have

$$\begin{aligned} \eta^{(+)a} &= i\,\{\,\sigma_2\otimes\sigma_1\,,\; -\sigma_2\otimes\sigma_3\,,\; I\otimes\sigma_2\,\} \\ \eta^{(-)a} &= i\,\{\,\sigma_1\otimes\sigma_2\,,\; \sigma_2\otimes I\,,\; -\sigma_3\otimes\sigma_2\,\} \,. \end{aligned} \tag{4}$$

where the σ's are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

satisfying the algebra $\sigma_i \sigma_j = \delta_{ij} - i\epsilon_{ijk}\sigma_k$.

One may therefore write a Dirac equation:

$$\left(\delta_{\mu\nu} D_4 + \eta^{(\pm)a}_{\mu\nu} D_a\right)\phi_\nu = 0 \quad , \quad a = 1,2,3. \tag{5}$$

for a four-component spinor ϕ_ν.

Now, as Belavin and Zakharov [16] realized, the (anti) self-duality equations follow as integrability conditions for (5) if we seek solutions of the form

$$\phi = \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \otimes \begin{pmatrix} 1 \\ i \end{pmatrix} \psi(\lambda, x) \quad ,$$

where $\psi(\lambda, x)$ is a matrix in the group space.

(ii) We note that representations of gamma matrices satisfying $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$, and having the above properties of reality and antisymmetry, recur only for the eight-dimensional gamma matrices of SO(7). (Apart from the trivial two dimensional case of $i\sigma_2$, when the analogue of (3), $(\sigma_2)_{ij} F_{ij} = 0$, is equivalent to the zero-curvature condition $F_{ij} = 0$). One set of seven real antisymmetric anticommuting 8×8 matrices with the negative of the identity matrix as the square is given by:

$$\begin{aligned}
\lambda^1 &= \sigma_1 \otimes i\sigma_2 \otimes \sigma_1 \\
\lambda^2 &= i\sigma_2 \otimes \sigma_1 \otimes \sigma_3 \\
\lambda^3 &= i\sigma_2 \otimes I \otimes \sigma_1 \\
\lambda^4 &= i\sigma_2 \otimes \sigma_3 \otimes \sigma_3 \\
\lambda^5 &= \sigma_3 \otimes i\sigma_2 \otimes \sigma_1 \\
\lambda^6 &= I \otimes i\sigma_2 \otimes \sigma_3 \\
\lambda^7 &= I \otimes I \otimes i\sigma_2
\end{aligned} \qquad \begin{aligned} \{\lambda^a, \lambda^b\} &= -2\delta^{ab} \\ \lambda^2 &= -\mathbb{1}. \end{aligned} \tag{6}$$

We now observe that the analogue of (3), viz.

$$\lambda^a_{ij} F_{ij} = 0 \qquad (i,j = 1, \ldots, 8). \tag{7}$$

is a set of seven equations (for the seven unknown gauge potentials) which imply the full eight dimensional Yang-Mills equations as a consequence of the Bianchi identities, as may be checked directly by writing out the seven equations (7) using the explicit representation (6):

$$
\begin{aligned}
F_{12} + F_{34} + F_{56} + F_{78} &= 0 \\
F_{13} + F_{42} + F_{75} + F_{68} &= 0 \\
F_{14} + F_{23} + F_{58} + F_{67} &= 0 \\
F_{16} + F_{25} + F_{83} + F_{47} &= 0 \\
F_{15} + F_{62} + F_{37} + F_{84} &= 0 \\
F_{17} + F_{82} + F_{53} + F_{46} &= 0 \\
F_{18} + F_{27} + F_{36} + F_{45} &= 0 \quad .
\end{aligned}
\tag{8}
$$

As these equations demonstrate, all eight indices appear on an equal footing, since each index appears once in each equation and each of the 28 components of the curvature two-form $F_{\mu\nu}$ appears in only one of the seven relations.

The above λ matrices may be constructed out of the structure constants of the octonions in the following fashion:

$$\lambda^{a}{}_{\mu\nu} \equiv C^{a}{}_{\mu\nu} + \delta_{a\mu}\delta_{\nu 8} - \delta_{a\nu}\delta_{\mu 8} \qquad ; (c.f.\ (1)) . \tag{9}$$

The totally antisymmetric C_{abc}'s determine the algebra of the octonions or Cayley numbers (see e.g. [94,100]) :

$$e_a e_b = -\delta_{ab} + C_{abc} e_c \qquad a = 1, \ldots\ldots, 7 .$$

where e_a are the imaginary octonions ($e_8 = 1$).

For the explicit realization (6) we obtain

$$1 = C_{127} = C_{145} = C_{136} = C_{235} = C_{264} = C_{347} = C_{567} \tag{9'}$$

(all others zero).

Eq.(9) yields the alternative form for these seven equations:

$$F_{8a} = \frac{1}{2} C_{abc} F_{bc} \tag{10}$$

(from (7)); a relation very similar to the four dimensional $F_{4a} = \frac{1}{2}\epsilon_{abc}F_{bc}$.

We note that these equations (8) set seven of the 28 pieces of the curvature to zero, leaving 21 pieces undetermined. In other words, the field strengths belong to a 21-dimensional representation of some subgroup

H of SO(8), contained in the decomposition of the adjoint of SO(8) under the breaking to the subgroup H. It is already clear that these equations do not possess the full SO(8) symmetry of the space-time; and that the only thing which allows the self-duality equations to preserve the full SO(4) symmetry is the fact that SO(4) is not simple. (All other SO(d) groups are simple). The subgroup of relevance for the seven equations here is clearly either Spin(7) (the covering group of SO(7), [98]) or SO(7), whose isomorphic Lie algebras are generated by the 21 bilinear combinations of the λ's :

$$\lambda^{ab} \equiv \tfrac{1}{2}\left[\lambda^a, \lambda^b\right] .$$

These, together with the seven λ^a's (the basis elements of the Clifford algebra C_7), form the Lie algebra of SO(8); i.e. $\{\lambda^a, [\lambda^a, \lambda^b]\}$ is a complete set of 8×8 antisymmetric matrices [95,97] , and decomposing any antisymmetric 8×8 matrix $A_{\mu\nu}$ in the form :

$$A_{\mu\nu} = b_a \lambda^a_{\mu\nu} + \tfrac{1}{2} c_{ab} \lambda^{ab}_{\mu\nu}$$

clearly yields the decomposition, $\underline{28} = \underline{7} + \underline{21}$, of the adjoint representation of SO(8) under its breaking to SO(7).

In order to understand the equations (7,8,10) we now go over to a manifestly eight (spatial) dimensional notation. We consider the gamma matrices acting on SO(8) spinors given in a chiral representation by

$$\gamma^8 = \begin{pmatrix} 0 & 1_8 \\ 1_8 & 0 \end{pmatrix} , \quad \gamma^a = \left(\begin{array}{c|c} 0 & i\lambda^a \\ \hline -i\lambda^a & 0 \end{array}\right) , \quad a = 1, \ldots 7 .$$

γ^μ, $\mu = 1, \ldots\ldots, 8$ are eight 16×16 matrices satisfying $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$.

Noting that

$$\gamma^9 = \begin{pmatrix} -\prod\limits_{1}^{7} \lambda^a & 0 \\ 0 & \prod\limits_{1}^{7} \lambda^a \end{pmatrix} = \begin{pmatrix} 1_8 & 0 \\ 0 & -1_8 \end{pmatrix} ,$$

and that $\gamma^{\mu\nu} \equiv \tfrac{1}{2}[\gamma^\mu, \gamma^\nu]$ has components

$$\gamma^{8a} = \begin{pmatrix} -\lambda^a & 0 \\ 0 & \lambda^a \end{pmatrix} , \quad \gamma^{ab} = \begin{pmatrix} -\tfrac{1}{2}[\lambda^a, \lambda^b] & 0 \\ 0 & -\tfrac{1}{2}[\lambda^a, \lambda^b] \end{pmatrix} ,$$

we see that $\tfrac{1}{2}\left(1 \overset{+}{(-)} \gamma^9\right)$ projects the top left (bottom right)-hand block of

SO(8) generators ; i.e. the left and right-handed spinors transform independently, and together with the vector representation form the three 8 dimensional representations of SO(8). The two eight dimensional blocks in γ^a above (each the negative of the other) are just the two inequivalent irreducible faithful representations of the gamma matrices of SO(7) (denoted by λ here) which provide two inequivalent embeddings of Spin(7) into SO(8) : corresponding to the little groups of constant left or right handed spinors.

Now we consider a Dirac equation in analogy to (5):

$$\not{D}\pi\phi = (\delta_{ij} D_8 + \lambda^a_{ij} D_a)\pi_j \phi = 0 \quad , (\pi \text{ a constant spinor}) \quad (11)$$

Then,

$$\not{D}\not{D}\pi\phi = (D^2 + \tfrac{1}{2}[\lambda^8,\lambda^a] F_{8a} + \tfrac{1}{2}[\lambda^a,\lambda^b] F_{ab})\pi\phi = 0,$$

which is satisfied if

$$\lambda^{\mu\nu}_{ij} F_{\mu\nu} \equiv (\lambda^a_{ij} F_{a8} + [\lambda^a,\lambda^b]_{ij} F_{ab}) = 0 \quad , \mu,\nu = a,b,8. \quad (12)$$

a set of 28 equations, which clearly implies that $F_{\mu\nu} = 0$.

Indeed, (11) itself implies that $D_\mu\phi = 0$, since

$$\pi^T_i \lambda^a_{ij} (\delta_{jk} D_8 + \lambda^b_{jk} D_b)\pi_k \phi = 0$$

implies, by the antisymmetry of λ^a that

$$\pi^T_i \lambda^a_{ij} \lambda^b_{jk} \pi_k D_b \phi = 0 ,$$

which, by the antisymmetry of λ^{ab}, directly yields $D_b\phi = 0$.

However, we may note that the 8×8 antisymmetric matrix $(\lambda^{\mu\nu})_{ij}$ in (12) is a matrix with each row, and each column, providing a representation of the λ's. In other words, if we fix j, by writing

$$(\lambda^{\mu\nu})_{ij}\eta_j = (M^{\mu\nu})_i \quad , (\eta \text{ a constant 8-spinor}) \quad (13)$$

then the $(M^{\mu\nu})_i$ in addition to the $(\lambda_{ij})^a$ form complete bases of the Clifford algebra C_7. The dual role of the two eight dimensional spaces is clear here; and is a consequence of the famous triality amongst the three eight dimensional representations of SO(8) [101].

In terms of the SO(8) gamma matrices, we clearly have

$$(M^{\mu\nu})_i = (\gamma^{\mu\nu})_{ij}\eta_{Lj} \equiv (\gamma^{\mu\nu}\eta)_i \quad ; \quad i,j,\mu,\nu = 1,\dots,8. \quad (14)$$

where η_L is a constant left-handed unit spinor of SO(8), $\eta^T\eta = 1$. (We use the fact that the spinors, just like the γ's, may be chosen to be real for SO(8)). Now, since these seven 8×8 matrices (14) form an alternative representation of the λ^a's, we may use them in place of the λ^a's in (7), obtaining

$$\gamma^{\mu\nu}\cdot\eta_L \; F_{\mu\nu} = 0 \quad , \tag{15}$$

an equivalent way of writing those seven equations.

Eq.(15) has seven components since the component in the direction of η: $\eta^T\gamma^{\mu\nu}\eta \; F_{\mu\nu}$, clearly vanishes identically because of the antisymmetry of $\gamma^{\mu\nu}_{AB}$, which implies that $\eta^T_A\gamma^{\mu\nu}_{AB}\eta_B = 0$.
Now, (15) implies that

$$\begin{aligned} 0 &= \eta^T_C \gamma^{\rho\sigma}_{CA} \gamma^{\mu\nu}_{AB} \eta_B \; F_{\mu\nu} \\ &= \left(\eta^T_C \gamma^{\rho\sigma\mu\nu}_{CB} \eta_B + \left(\delta^{\sigma\mu}\delta^{\rho\nu} - \delta^{\rho\mu}\delta^{\sigma\nu}\right)\right) F_{\mu\nu} \end{aligned}$$

i.e.

$$\tfrac{1}{2}\,\eta^T\gamma^{\rho\sigma\mu\nu}\eta \; F_{\mu\nu} = F_{\rho\sigma} \quad , \tag{16}$$

where

$$\gamma^{\mu\nu\rho\sigma} \equiv \frac{1}{4!}\,\gamma^{[\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma]} \tag{17}$$

is given by,

$$\gamma^{8abc} = \begin{pmatrix} \lambda^a\lambda^b\lambda^c & 0 \\ 0 & -\lambda^a\lambda^b\lambda^c \end{pmatrix}, \quad \gamma^{abcd} = \begin{pmatrix} \lambda^a\lambda^b\lambda^c\lambda^d & 0 \\ 0 & \lambda^a\lambda^b\lambda^c\lambda^d \end{pmatrix},$$

where a,b,c,d are all different, and take values 1,.......,7 ; and since $\;{}^*\gamma^{\mu\nu\rho\sigma} = \frac{1}{4!}\,\epsilon^{\mu\nu\rho\sigma\alpha\beta\gamma\delta}\,\gamma^{\alpha\beta\gamma\delta} = \gamma^9\gamma^{\mu\nu\rho\sigma}$,
i.e. the left-and right-handed pieces in (17) are dual to each other, we may equally consider a right-handed spinor $\eta_R \equiv \eta = \gamma^9\eta$ in the above equations (14-16). Equation (16) is in fact equivalent to (14), which we may prove by showing that it in turn implies (14). We use the completeness relation for the 28 antisymmetric matrices $(\gamma^{\mu\nu})_{AB}$ [95] :

$$\gamma^{\mu\nu}_{AB}\gamma^{\mu\nu}_{CD} = 8\left(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}\right) \quad ,$$

which yields the identity

$$\lambda^{ab}_{AB}\,\eta_B\,\eta^T_C\,\lambda^{ab}_{CD} = 8\left(\eta_D\eta^T_A - \delta_{AD}\right) . \tag{18}$$

Eq.(16) may be written

$$\left(\eta_C^T \, \gamma^{\mu\nu}_{CE} \, \gamma^{\rho\sigma}_{ED} \, \eta_D - \left(\delta^{\nu\rho}\delta^{\mu\sigma} - \delta^{\mu\rho}\delta^{\nu\sigma}\right)\right) F_{\rho\sigma} = 2 F_{\mu\nu}$$

i.e.
$$\eta_C^T \, \gamma^{\mu\nu}_{CE} \, \gamma^{\rho\sigma}_{ED} \, \eta_D \, F_{\rho\sigma} = 0$$

Multiplying on the left by $\gamma^{\mu\nu}_{FG}\,\eta_G$, we obtain

$$\gamma^{\mu\nu}_{FG} \, \eta_G \, \eta_C^T \, \gamma^{\mu\nu}_{CE} \, \gamma^{\rho\sigma}_{ED} \, \eta_D \, F_{\rho\sigma} = 0 ,$$

which yields $\gamma^{\rho\sigma}_{FD}\,\eta_D\,F_{\rho\sigma} = 0$ on using (18).

Therefore

$$F_{\mu\nu} = \tfrac{1}{2}\, \eta_L^T \, \gamma^{\mu\nu\rho\sigma} \eta_L \, F_{\rho\sigma} \iff \gamma^{\rho\sigma} \eta_L \, F_{\rho\sigma} = 0 . \tag{19}$$

We therefore have the completely antisymmetric object

$$T_{\mu\nu\rho\sigma} = \eta^T \, \gamma^{\mu\nu\rho\sigma} \eta , \tag{20}$$

which mimics the four-dimensional duality operator in the sense that it maps the space of two-forms to itself. It is clear that since η in (20) may be chosen to be either left- or right-handed, corresponding to the self or anti-self-dual part of T i.e. $\frac{1}{2}\left(T_{\mu\nu\rho\sigma} \pm {}^{*}T_{\mu\nu\rho\sigma}\right)$, T transforms as one of the two 35 dimensional antisymmetric tensor representations of SO(8). Under a breaking to Spin(7), one of the $\underline{35}$'s is reduced to $\underline{1} + \underline{27} + \underline{7}$, clearly allowing a Spin(7)-invariant tensor T, since the decomposition contains a singlet; confirming that Spin(7) is the stability group of our equations. We observe that the form (20) explicitly demonstrates that the $\underline{35}$ is the one contained in $\underline{8}_s \otimes \underline{8}_s = \underline{1} + \underline{28} + \underline{35}$, where the $\underline{35}$ is the symmetric, traceless part of the tensor product, and s denotes a spinor representation. Under Spin(7) one of the spinor $\underline{8}$'s of SO(8) decomposes into $\underline{8}_s = \underline{1} + \underline{7}$, yielding a decomposition of the corresponding $\underline{35}$ with a singlet; whereas under SO(7) both the 8 dimensional spinor representations of SO(8) remain irreducible, (only the vector $\underline{8}_v = \underline{1} + \underline{7}$).

Thus far we have identified the components of $F_{\mu\nu}$ in the 21 dimensional orbit into which the $\underline{28}$ of SO(8) splits under the action of Spin(7). We identify the orthogonal 7 components of $F_{\mu\nu}$ by noting that (19) may be generalized thus:

$$\lambda F_{\mu\nu} = \tfrac{1}{2} \, T_{\mu\nu\rho\sigma} \, F_{\rho\sigma} , \tag{21}$$

(where $\lambda = 1$ for the seven equations above), since for all possible eigenvalues of T, the antisymmetry of T means that the Yang-Mills equations are satisfied as a consequence of the Bianchi identities in a non-trivial fashion. (For λ not an eigenvalue of T, the relation (21) trivializes the theory, i.e. $F_{\mu\nu} = 0$).

We now note that

$$\left(\eta^T \gamma^{\mu\nu\rho\sigma} \eta\right)\left(\eta^T \gamma^{\rho\sigma\alpha\beta} \eta\right)$$

$$= \left(\eta^T \gamma^{\mu\nu}\gamma^{\rho\sigma}\eta - \left(\delta^{\nu\rho}\delta^{\mu\sigma} - \delta^{\nu\sigma}\delta^{\mu\rho}\right)\right)\left(\eta^T\gamma^{\rho\sigma}\gamma^{\alpha\beta}\eta - \left(\delta^{\sigma\alpha}\delta^{\rho\beta} - \delta^{\sigma\beta}\delta^{\rho\alpha}\right)\right)$$

$$= 8\,\eta^T\gamma^{\mu\nu}\left(\eta\eta^T - 1\right)\gamma^{\alpha\beta}\eta + 4\eta^T\gamma^{\mu\nu}\gamma^{\alpha\beta}\eta + 2\left(\delta^{\mu\alpha}\delta^{\nu\beta} - \delta^{\nu\alpha}\delta^{\mu\beta}\right), \text{ using (18);}$$

$$= -4\,\eta^T\gamma^{\mu\nu}\gamma^{\alpha\beta}\eta + 2\left(\delta^{\mu\alpha}\delta^{\nu\beta} - \delta^{\nu\alpha}\delta^{\mu\beta}\right) \quad \text{, since } \eta^T\gamma^{\mu\nu}\eta = 0\,;$$

$$= -4\,\eta^T\gamma^{\mu\nu\alpha\beta}\eta - 6\left(\delta^{\nu\alpha}\delta^{\mu\beta} - \delta^{\mu\alpha}\delta^{\nu\beta}\right).$$

Therefore,

$$\tfrac{1}{2}\, T_{\mu\nu\rho\sigma}T_{\rho\sigma\alpha\beta} + 2\,T_{\mu\nu\alpha\beta} - 3\left(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right) = 0\,, \qquad (22)$$

and the other value of λ which yields non-trivial relations among the $F_{\mu\nu}$'s may be deduced to be -3, since writing (16) as $(T - \lambda\mathbb{1}).F = 0$, where $\mathbb{1}$ denotes $(\delta^{\mu\nu}\delta^{\nu\rho} - \delta^{\mu\rho}\delta^{\nu\sigma})$, we obtain

$$0 = \left(\tfrac{1}{2}T^2 - \lambda T\right)\cdot F = \left((-2-\lambda)T + 3\,\mathbb{1}\right)\cdot F \quad \text{, using (22);}$$

$$= \left((-2-\lambda)\lambda + 3\right)\mathbb{1}\cdot F$$

which yields $\lambda = 1\,,\ -3$.

Using the octonion structure constants given above, T may be written

$$T_{\mu\nu\rho\sigma} = \sum_{(\alpha\beta\gamma\delta)} \epsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta} \qquad (23)$$

where the $(\alpha\beta\gamma\delta)$ runs over the set

$$\left\{ \begin{array}{l} 1234,\ 1256,\ 1278,\ 1357,\ 1386,\ 1476,\ 1485, \\ 5678,\ 3478,\ 3456,\ 2468,\ 2475,\ 2385,\ 2376 \end{array} \right\}, \qquad (24)$$

which is clearly the self-dual part of

$$T_{\mu\nu\rho\sigma} = \tfrac{1}{4}\,\epsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta}\,V_\delta\,C_{\alpha\beta\gamma}\,; \qquad (25)$$

$$V_\delta = (1, 0, \ldots\ldots, 0)\,, \qquad \alpha,\beta,\gamma = 2,\ldots\ldots,8\,.$$

The duality properties of T now imply that if we pick out a preferred direction, say $\mu = 8$, we may write $T_{8\nu\rho\sigma} = C_{\nu\rho\sigma}$;

clearly yielding the previously obtained form (10); and also a remarkable representation for the octonion stucture constants:

$$c_{\nu\rho\sigma} = \eta^T \gamma_{\nu\rho\sigma} \eta \quad .$$

Inserting $\lambda = -3$ in (21) we obtain the equality of each term in each row of eq.(8); i.e. 21 equations of the form

$$F_{12} = F_{34} = F_{56} = F_{78} \qquad \text{etc.,} \tag{26}$$

which transform as the $\underline{7}$ of Spin(7).

We note that the Spin(7) invariant 4-form (23) has also recently been discussed in the mathematical literature [99] .

(iiia) We have now clearly identified the essential features which allow one to write down non-trivial algebraic relations of the generic form

$$\frac{1}{2} T_{\mu\nu\rho\sigma} F_{\rho\sigma} = \lambda F_{\mu\nu} \quad , \tag{27}$$

which clearly imply, via the Bianchi identities, that the Yang-Mills equations are satisfied. For d = 4 , T is essentially unique : $T_{\mu\nu\rho\sigma} = \epsilon_{\mu\nu\rho\sigma}$, which has eigenvalues ∓ 1 , yielding the usual (anti) self-duality equations. For other values of λ , it is obvious that $F_{\mu\nu} = 0$. For any $d > 4$, as we have already emphasized, T cannot be invariant under SO(d), since the only invariant totally antisymmetric object is the d-dimensional duality operator. However, as we have seen, T will be invariant under some subgroup H of SO(d). It is this which gives us a handle with which to attempt to classify interesting relations of the form (27). Except for the already discussed (and exceptional) eight dimensional case, T belongs to an irreducible representation of SO(d) of dimension $\binom{d}{4}$, since the set of all 4-forms in d-dimensional space is a vector space of this dimensionality. We need to investigate the breaking of the $\binom{d}{4}$ representation into representations of H. If the decomposition does not contain a singlet, it is clear that an H-invariant set of equations of the form (27) does not exist. (Apart from the trivial G-invariant equations : $F_{\mu\nu} = 0$). On the other hand, if the decomposition contains a singlet, it is clear that we may construct an H-invariant T and find its eigenvalues λ . Then, according to (27), given a non-trivial λ, the pieces of the field strength corresponding to the other λ's vanish. As a result, the adjoint representation of SO(d) according to which $F_{\mu\nu}$ transforms splits into orbits under the action of H ; the curvatures in each orbit corresponding to the same eigenvalue. To illustrate this, we proceed to give further examples of H-invariant sets of equations. We shall consider all maximal subgroups H for dimensions 5 to 8, and we shall construct invariant T's in those cases where this is possible. A list of maximal subgroups of SO(d) and the decompositions under them of the relevant representations of SO(d) [96] is given in the appendix.

(iiib) In five dimensions, H is clearly the SO(4) leaving a constant vector, n_μ say, invariant; and if n_μ is a unit vector,

$$T_{\mu\nu\rho\sigma} = \epsilon_{\mu\nu\rho\sigma\gamma}\, n_\gamma \tag{28}$$

with $\lambda = \mp 1$. This case just yields the usual self-duality equations in the 4-dimensional subspace orthogonal to n, together with $n_\mu F_{\mu\nu} = 0$. These equations, rotated to Minkowski 5-space, for particular choices of n, yield the Bogomolny equations for a uniformly moving set of monopoles, or the equations for the Julia-Zee dyon.

Analogously with (28), for any dimension d, we may clearly have an H-invariant T which yields the self-duality equations in some four dimensional subspace which is projected out of the d-dimensional Euclidean space by an orthonormal set of (d-4) unit vectors, along each of which the curvature vanishes, and with H = SO(4)⊗ SO(d-4). For instance, for d = 6,

$$T_{\mu\nu\rho\sigma} = \epsilon_{\mu\nu\rho\sigma\alpha\beta}\, m_\alpha n_\beta \qquad , \text{ m,n orthonormal;} \tag{29}$$

is clearly SO(4)⊗SO(2) invariant, yielding with $\lambda = \pm 1$, the (anti) self-duality equations in the 4-space orthogonal to m and n, together with

$$n_\mu F_{\mu\nu} = 0 = m_\mu F_{\mu\nu} \ .$$

Similarly, for d = 7,

$$T_{\mu\nu\rho\sigma} = \epsilon_{\mu\nu\rho\sigma\alpha\beta\gamma}\, m_\alpha n_\beta k_\gamma \qquad , \text{ m,n,k orthonormal;} \tag{30}$$

is an obvious $SO(4)\otimes SO(3)$-invariant, and the analogous $SO(4)\otimes SO(4)$-invariant object for $d = 8$ may clearly be written

$$T_{\mu\nu\rho\sigma} = \alpha\, \epsilon_{\mu\nu\rho\sigma 1234} + \beta\, \epsilon_{\mu\nu\rho\sigma 5678} \tag{31}$$

(choosing an obvious complete orthonormal set of vectors), with $\lambda = \pm\alpha\pm\beta$, reducing to self- and anti-self-duality in the appropriate variables. The case with $\alpha = +1$, $\beta = -1$ is precisely the one considered by Witten [72] in his discussion of the full Yang-Mills equations. We may remark that since all cases of the type (28-30), for arbitrary dimension, are just (anti) self-duality relations in the four dimensional subspaces orthogonal to the orthonormal set of (d-4) vectors, and zero curvatures elsewhere, all these cases are clearly integrable; and the linear systems are just staightforward generalizations of the four dimensional case. The case (31) is also clearly integrable, since it corresponds to a direct product of two sets of (anti) self-duality relations. The integrability of Witten's case ($\alpha = 1$, $\beta = -1$) has been discussed by Forgacs et al [102] . Since all these cases (28-31) are effectively four dimensional, they are not very interesting. More interesting and nontrivial are the cases displaying an octonionic structure, which for $d < 8$ may be obtained by dimensional reduction from the eight-dimensional equations. We shall see that such cases exhaust all further relations invariant under maximal subgroups of SO(d).

(iiic) First we consider the $d = 7$ case, where apart from the already-discussed $SO(4)\otimes SO(3)$ case, G_2 is the only other maximal subgroup under which T (a $\underline{35}$ of SO(7)) contains a singlet (see appendix). From (10), deleting the index 8, we see that

$$c_{abc} F_{bc} = 0 \qquad , a,b = 1,\ldots\ldots,7 ; \tag{32}$$

are manifestly G_2-invariant equations, since G_2 is the automorphism group of the octonions. We clearly have the G_2 invariant :

$$T_{\mu\nu\rho\sigma} = \frac{1}{3!}\, \epsilon_{\mu\nu\rho\sigma\alpha\beta\delta}\, c_{\alpha\beta\delta} \quad , \tag{33}$$

with $\lambda = 1$, -3 as before, as may now be deduced from the identity [100]

$$c_{abc}\, c_{cde}\, c_{efa} = 3\, c_{bdf} \quad , \tag{34}$$

as a consequence of the further identities:

$$\frac{1}{3!}\,\epsilon_{\mu\nu\rho\sigma\alpha\beta\delta}\,C_{\alpha\beta\delta} = (\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\rho\nu}\delta_{\mu\sigma}) + C_{\mu\nu\gamma}C_{\rho\sigma\gamma} \tag{35}$$

and

$$C_{\alpha\beta\gamma} = -\frac{1}{4!}\,\epsilon_{\alpha\beta\gamma\delta\mu\nu\rho}\,C_{\delta\mu\epsilon}\,C_{\nu\rho\epsilon}\ , \tag{36}$$

which correspond to (22). Equation (27) with T given by (33) is clearly equivalent to (32) since

$$F_{\mu\nu} = \frac{1}{2}\,\frac{1}{3!}\,\epsilon_{\mu\nu\rho\sigma\alpha\beta\delta}\,C_{\alpha\beta\delta}\,F_{\rho\sigma}$$

may be written

$$0 = (\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} - \frac{1}{3!}\,\epsilon_{\mu\nu\rho\sigma\alpha\beta\delta}\,C_{\alpha\beta\delta})\,F_{\rho\sigma}$$

$$= -\,C_{\mu\nu\gamma}\,C_{\rho\sigma\gamma}\,F_{\rho\sigma} \qquad \text{, by (35);}$$

which implies, by (36), that

$$C_{\rho\sigma\gamma}\,F_{\rho\sigma} = 0\ .$$

As before, the relations orthogonal to (32), of which there are 14 here, may be obtained by insisting on the equality of the three terms in each of the seven components of $C_{abc}\,F_{bc}$.

We now return to d = 6 , where apart from the above considered $SO(4)\otimes SO(2)$, the only maximal subgroup leaving T invariant is $SU(3)\otimes U(1)/Z_3$, under which the adjoint of SO(6) (according to which both T and F transform in 6 dimensions), has the decomposition:

$$\underline{15} = (\underline{\bar{3}}_2 + \underline{3}_{-2}) + \underline{1}_0 + \underline{8}_0 \tag{37}$$

where in $\underline{a}_b$, a is the SU(3) dimension, and b the U(1) quantum number. (We use the notation of [96]). Noting that the SU(3) subgroup can be imbedded in G_2; and that it is in fact the subgroup of G_2 which leaves any one of the imaginary basis elements of the octonions invariant [94] (i.e. it is the automorphism group of the multiplication rules among six of the seven imaginary octonion units), we may make the identification

$$T_{\mu\nu\rho\sigma} = \frac{1}{2}\,\epsilon_{\mu\nu\rho\sigma\alpha\beta}\,C_{\alpha\beta 7} \quad , \ \alpha,\beta = 1,\ldots\ldots,6\,. \tag{38}$$

We note that since we are considering the SU(3) here as a subgroup of G_2, which contains only real representations, the pair $(\underline{\bar{3}}_2 + \underline{3}_{-2})$in (37) need to be considered together for our purposes, as a real six dimensional representation. From the explicit representation for the octonion structure constants displayed above (9'), we see that (38) has the alternative representation

$$T_{\mu\nu\rho\sigma} \;=\; \epsilon_{\mu\nu\rho\sigma}(y\bar{y} + z\bar{z} + t\bar{t}) \tag{39}$$

where we introduce complex variables

$y = x_1 + ix_2$, $z = x_3 + ix_4$, $t = x_5 + ix_6$ and their complex conjugates $(\bar{y}, \bar{z}, \bar{t})$, in terms of which the euclidean metric is $dy\,d\bar{y} + dz\,d\bar{z} + dt\,d\bar{t}$. (We use the convention $\epsilon_{y\bar{y}z\bar{z}t\bar{t}} = 1$).

Now, the octet in (37), corresponding to a set of seven equations, may be obtained by trivial dimensional reduction from (32) (just by deleting terms with the index 7). This set, as before, corresponds to $\lambda = 1$. We now note that because of the decomposition (37), the orthogonal pieces of the curvature no longer live in the same orbit ; a degeneracy in the eigenvalue equation has been split by the dimensional reduction. The 21 equations (26) in eight dimensions would yield 8 equations if the indices 7 & 8 were deleted, whereas the decomposition (37) means that we have sets of 14 and 9 equations corresponding to the singlet and $(\underline{3}+\bar{\underline{3}})$ pieces respectively. Indeed, it is easy to check that the eight equations obtained by dimensional reduction of (26) do not even satisfy the six dimensional Yang-Mills equations. (This situation, of on-shell constraints dimensionally reducing to off-shell relations, is familiar in supersymmetric gauge theories). Corresponding to the $\underline{1}$ and $\underline{3} + \bar{\underline{3}}$, T has eigenvalues -2 and -1 respectively, and the corresponding relations are :

$\underline{1}$:

$$\left.\begin{array}{l} F_{y\bar{y}} = F_{z\bar{z}} = F_{t\bar{t}} \\ \text{all other (twelve) curvatures zero} \end{array}\right\} \quad \text{14 equations} \tag{40}$$

$\underline{3}+\bar{\underline{3}}$:

$$\left.\begin{array}{l} F_{y\bar{y}} + F_{z\bar{z}} + F_{t\bar{t}} = 0 \\ F_{y\bar{z}} = F_{y\bar{t}} = F_{z\bar{t}} = 0 \end{array}\right\} \quad \text{9 equations} \tag{41}$$

(iiid) We now return to d = 8 , where SO(8) has four maximal subgroups leaving $T_{\mu\nu\rho\sigma}$ invariant. They are $SO(4)\otimes SO(4)$ and Spin(7), which we have already considered; and $(SU(4)\otimes U(1))/Z_4$ and $(Sp(4)\otimes SU(2))/Z_2$, which we now discuss. We first consider the $SU(4)\otimes U(1)/Z_4$ invariant case. The decomposition of the relevant representations of SO(8) are :

$$\begin{aligned} \underline{28} &= \underline{1}_0 + \underline{15}_0 + (\underline{6}_2 + \underline{6}_{-2}) \\ \underline{35}(\text{self-dual}) &= \underline{1}_0 + (\underline{1}_4 + \underline{1}_{-4}) + (\underline{6}_2 + \underline{6}_{-2}) + \underline{20}_0 \\ \underline{35}(\text{anti-dual}) &= \underline{15}_0 + (\underline{10}_2 + \underline{10}_{-2}) \end{aligned} \tag{42}$$

in the notation of (37).

We now note the similar form of the decomposition of the adjoints in the two cases (42) and (37) , and analogously to the explicitly $SU(3)\otimes U(1)/Z_3$-invariant (39), we may, introducing a fourth complex variable $w = x_7 + ix_8$, write down the $SU(4)\otimes U(1)/Z_4$-invariant; the singlet piece of the self-dual $\underline{35}$:

$$\begin{aligned} T_{\mu\nu\rho\sigma} &= \epsilon_{\mu\nu\rho\sigma}(q\bar{q}z\bar{z} + q\bar{q}t\bar{t} + q\bar{q}w\bar{w} + z\bar{z}t\bar{t} + z\bar{z}w\bar{w} + t\bar{t}w\bar{w}) \\ &= \sum_{(\alpha\beta\gamma\delta)} \epsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta} \quad , \end{aligned} \tag{43}$$

where $(\alpha\beta\gamma\delta)$ runs over the set:

$\{1234,\ 1256,\ 1278,\ 3456,\ 3478,\ 5678\}$,

which is clearly self-dual.

In a more covariant notation, (43) may be expressed in terms of the octonion structure constants:

$$T_{\mu\nu\rho\sigma} = \frac{1}{4}\, \epsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta}\, c_{\alpha\beta\kappa}\, v_\kappa\, c'_{\gamma\delta\varepsilon}\, u_\varepsilon \tag{44}$$

where v, u are constant orthonormal vectors, and the indices on $c_{\alpha\beta\kappa}$, $c'_{\gamma\delta\varepsilon}$ span different 7 dimensional subspaces, e.g. taking the non-zero components of $c_{\alpha\beta\kappa}$ to be those given in (9'), we may choose

$$c'_{\gamma\delta\varepsilon} = 1 \quad \text{for} \quad \gamma\delta\varepsilon = 782,\ 763,\ 754,\ 853,\ 846,\ 342,\ 562. \tag{45}$$

Then, an appropriate choice of the vectors u and v , viz.,

$$\begin{aligned} v_\kappa &= 1 \ \text{ for } \kappa = 7 \qquad , \qquad & u_\varepsilon &= 1 \ \text{ for } \varepsilon = 2 \\ &= 0 \ \text{ otherwise} & &= 0 \ \text{ otherwise} \end{aligned}$$

yields the previous form (43).

Further, we may recall that the self-dual $\underline{35}$ is the symmetric, traceless part of $\underline{8}_s\otimes\underline{8}_s$ (see appendix), and since the SO(8) spinor decomposes into an SU(4) vector under the breaking to $SU(4)\otimes U(1)/Z_4$, viz.,

$$\underline{8} = \underline{1}_2 + \underline{1}_{-2} + \underline{6}_0 ,$$

we may choose the spinor in (20) to be :

$$\eta_a = \epsilon_A \chi_i = \epsilon_A \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \qquad \text{where } \begin{array}{l} a = 1,\ldots\ldots,8; \\ A = 1,\ldots\ldots,6; \\ i = 7,8. \end{array}$$

and where ϵ_A is a constant 6-vector. Choosing this to be of the form $\epsilon_A = (1,0,\ldots\ldots,0)$, and inserting in (20) (using the representation of gamma matrices used there), we again obtain (43). The eigenvalues of (43) are $\lambda = -3,\ 1,\ -1$ corresponding, respectively, to the $\underline{1}$, $\underline{15}$ and $\underline{12}$ pieces of the $\underline{28}$. (Note that since T is traceless, $1(-3)+15(1)+12(-1) = 0$). For these eigenvalues, (21) reduces to

$\underline{1}$: 27 equations : $\begin{cases} F_{y\bar{y}} = F_{z\bar{z}} = F_{t\bar{t}} = F_{w\bar{w}} , \\ \text{the other 24 curvatures vanish.} \end{cases}$

$\underline{15}$: 13 equations : $\begin{cases} F_{y\bar{y}} + F_{z\bar{z}} + F_{t\bar{t}} + F_{w\bar{w}} = 0 \\ F_{ab} = F_{\bar{a}\bar{b}} = 0 \quad ; \ a,b = y,z,t,w . \end{cases}$

$\underline{12}$: 16 equations : $F_{a\bar{b}} = 0 \quad ; \ a, b = y, z, t, w .$

We finally turn to the $Sp(4)\otimes SU(2)/Z_2$ case 103 , where the decompositions of the relevant SO(8) representations are [104] :

$$\begin{aligned} \underline{35}(\text{self-dual}) &= (\underline{14},\underline{1}) + (\underline{5},\underline{3}) + (\underline{1},\underline{5}) + (\underline{1},\underline{1}) \\ \underline{35}(\text{anti-dual}) &= (\underline{10},\underline{3}) + (\underline{5},\underline{1}) \\ \underline{28} &= (\underline{10},\underline{1}) + (\underline{5},\underline{3}) + (\underline{1},\underline{3}) \\ \underline{8} &= (\underline{5},\underline{1}) + (\underline{1},\underline{3}) \end{aligned}$$

Noting that $Sp(4)\subset SU(4)$, we may obtain the invariant T by generalizing the U(1) invariance of (44) (corresponding to rotations in the (v,u) subspace) to an SU(2), and projecting out the $Sp(4)\otimes SU(2)/Z_2$ -invariant object by using another constant vector w . We consider

$$w_\sigma \, \epsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta} \, A_{\mu\nu\rho} \tag{46}$$

with the three-form $A_{\mu\nu\rho} = \frac{1}{4} \epsilon_{\mu\nu\rho\, abcde}\, C_{bcf}\, C'_{deg}\, \epsilon_{afg}$,

where the indices on ϵ_{afg} span a fixed 3-dimensional subspace.

The invariant T is the self-dual part of (46). Choosing c_{bcf} and c'_{deg} as above (9', 45), $\epsilon_{afg} = \epsilon_{172} = 1$, and $w = (1, 0, \ldots\ldots, 0)$, we obtain:

$$\begin{aligned} T_{\alpha\beta\gamma\delta} &= \left(W_\sigma \epsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta} A_{\mu\nu\rho} + W_{[\alpha} A_{\beta\gamma\delta]} \right) \\ &= \sum_{(\mu\nu\rho\sigma)} \epsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta} \end{aligned} \tag{47}$$

where $(\mu\nu\rho\sigma)$ runs over the set :

$\{1265, 1287, 1243, 1674, 1537, 3487, 3465, 5687, 5238, 4268\}$

This has eigenvalues $\lambda = 1, -3, 5$ corresponding to the decomposition of the adjoint into its 15, 10 and 3 dimensional pieces respectively, and yielding sets of 13, 18 and 25 equations amongst the 28 components of F [105].

(iv) We conclude with some comments concerning the integrability of the original, and most interesting, set of 7 equations in 8 dimensions (8). We consider [106] the pair of quaternionic vector fields:

$$\begin{aligned} v_1 &= \gamma^2 \partial_y + i\,\partial_z + j\,\partial_w + k\,\partial_t + \gamma\left(-\gamma^{-2}\partial_{\bar y} - i\partial_{\bar z} - j\partial_{\bar w} - k\,\partial_{\bar t}\right) \\ v_2 &= \gamma^2 \partial_y - i\,\partial_z - j\,\partial_w - k\,\partial_t + \gamma\left(\gamma^{-2}\partial_{\bar y} - i\partial_{\bar z} - j\partial_{\bar w} - k\,\partial_{\bar t}\right) \end{aligned}$$

where i, j, k are the imaginary units of the quaternions and γ is a complex parameter.

Then, the curvature 2-form evaluated on these vectors is given by

$$\begin{aligned} F(v_1, v_2) = \; & \gamma\left(F_{y\bar y} + F_{z\bar z} + F_{w\bar w} + F_{t\bar t}\right) \\ & - \gamma^2\left[i\left(F_{yz} - F_{\bar w\bar t}\right) + j\left(F_{yw} + F_{\bar z\bar t}\right) + k\left(F_{yt} - F_{\bar z\bar w}\right)\right] \\ & + i\left(F_{\bar y\bar z} - F_{wt}\right) + j\left(F_{\bar y\bar w} + F_{zt}\right) + k\left(F_{\bar y\bar t} - F_{zw}\right) . \end{aligned}$$

The coefficients of γ, i, j, k, $\gamma^2 i$, $\gamma^2 j$, $\gamma^2 k$ are just the seven curvatures set to zero in (8). One may therefore think of v_1 and v_2 as vectors spanning a quaternionic plane on which the curvature vanishes.

Equivalently, defining

$$\begin{aligned} \nabla_1 \phi &= \left(D_{\gamma^2 y - \gamma^{-1}\bar y} + i\, D_{z - \gamma\bar z} + j\, D_{w - \gamma\bar w} + k\, D_{t - \gamma\bar t}\right)\phi \\ \nabla_2 \phi &= \left(D_{\gamma^2 y + \gamma^{-1}\bar y} - i\, D_{z + \gamma\bar z} - j\, D_{w + \gamma\bar w} - k\, D_{t + \gamma\bar t}\right)\phi \end{aligned}$$

(ϕ a quaternionic matrix)

the seven equations result from the vanishing of

$$(\nabla_1 \wedge \nabla_2)\,\phi$$

where $\wedge$ denotes noncommutative antisymmetrized outer multiplication.

We may also observe that defining two complex quaternionic one-forms:

$$dv_1 = (\gamma^2 dy - \gamma^{-1} d\bar{y}) + i(dz - \gamma d\bar{z}) + j(dw - \gamma d\bar{w}) + k(dt - \gamma d\bar{t})$$
$$dv_2 = (\gamma^2 dy + \gamma^{-1} d\bar{y}) - i(dz + \gamma d\bar{z}) - j(dw + \gamma d\bar{w}) - k(dt + \gamma d\bar{t}),$$

the coefficients of $\gamma, i, j, k, \gamma^2(i, j, k)$ in $dv_1 \wedge dv_2$ are precisely a basis for T-dual 2-forms (i.e. 2-forms satisfying $F_{\mu\nu} = \frac{1}{2} T_{\mu\nu\rho\sigma} F_{\rho\sigma}$); and one may think of $dv_1 \wedge dv_2$ as a T-dual 2-form with values in the Lie algebra of the complexified group of all quaternions. Thus, any connection form with curvature $F \sim dv_1 \wedge dv_2$ will automatically be a solution to the 7 equations (8) for this gauge group. This is to be compared with the BPST [19] case, where

$$A = \mathrm{Im} \frac{x\, d\bar{x}}{1 + |x|^2}, \quad x = x^1 + i x^2 + j x^3 + k x^4,$$

yields

$$F = (1 + |x|^2)^{-2}\, dx \wedge d\bar{x},$$

where the coefficients of i, j, k in $dx \wedge d\bar{x}$ provide a basis for self-dual 2-forms, thus giving an SU(2) instanton. However, for the present case, it is not clear how one is to integrate $dv_1 \wedge dv_2$.

We now note that if we dimensionally reduce from 8 to 4 dimensions by deleting the 5,6,7,8-subscripted terms, the seven equations (8) yield just anti-self duality. This was to be expected, however the construction for the 4-form T (20), remarkably, reduces to (since $\gamma^{\mu\nu\rho\sigma} = \gamma^5 \epsilon^{\mu\nu\rho\sigma}$). Since the (anti) self-duality equations are embedded in these 7 equations, it is clear that we may find integrability conditions for a sector of the theory described by these equations as long as sufficiently many additional constraints are imposed on the curvatures so as to effectively reduce the theory to four dimensions. We explicitly demonstrate this for the seven equations in six dimensions obtained by deleting terms with indices 7,8 in (8); i.e. the case corresponding to (38,39):

$$\begin{aligned} &F_{y\bar{y}} + F_{z\bar{z}} + F_{x\bar{x}} = 0 \\ &F_{xy} = 0 = F_{xz} = F_{yz}. \end{aligned} \tag{45}$$

We impose the extra constraints:

$$V_{\bar{a}} F_{a\bar{b}} = 0, \quad a, b = x, y, z. \tag{46}$$

where v is a constant vector. Then, on using (46), it is easy to show that equations (45) are implied by the integrability conditions for the linear system :

$$\nabla_1 \phi \equiv (D_x - \lambda v_{\bar{y}} D_{\bar{z}} + \lambda v_{\bar{z}} D_{\bar{y}})\phi = 0$$
$$\nabla_2 \phi \equiv (D_y + \lambda v_{\bar{x}} D_{\bar{z}} - \lambda v_{\bar{z}} D_{\bar{x}})\phi = 0$$
$$\nabla_3 \phi \equiv (D_z + \lambda v_{\bar{y}} D_{\bar{x}} - \lambda v_{\bar{x}} D_{\bar{y}})\phi = 0 \quad ,$$

which is comparable to the linear system of section 3.3 . Indeed, v here may be taken to be a space-time dependent complex Killing-vector field, in which case, the effective four-dimensional space would be curved.

The latter remark also applies to the orthonormal unit vectors of section (iiib) above. Indeed, if for instance, the vector n_μ in (28) is taken to be the radial vector $x_\mu/\sqrt{x^2}$, we obtain the self-duality equations on S^4 ; alternatively, if we choose $n_\mu = (x_i/\sqrt{x^2}, 0, 0)$, $\mu = 1,\ldots\ldots,5$, $i = 1,2,3$, the corresponding equations are easily seen to be self-duality relations over the four dimensional space $R^2 \times S^2$. This suggests the existence of non-trivial generalizations of the equations we have discussed in this chapter, corresponding to space-time dependent $T_{\mu\nu\rho\sigma}$'s.

Appendix

i) Properties of SO(n) representations. [96] .

a. Tensor products.

SO(8):

$$8_i \times 8_i = 1 + 28_A + 35_i \quad ; \quad i = v, s, c \; ; \; A = \text{antisymmetric} .$$

$$8_i \times 8_j = 8_k + 56_k \quad ; \quad (i, j, k \text{ cyclic})$$

$$35_i \times 28 = 28 + 35_i + 350 + 567_i$$

SO(7):

$$8 \times 8 = 1 + 7_A + 21 + 35$$

$$35 \times 21 = 7 + 21 + 35 + 105 + 189 + 378$$

SO(6):

$$4 \times 4 = 6 + 10$$

$$4 \times \bar{4} = 1 + 15$$

$$6 \times 6 = 1 + 15 + 20$$

b. Branching rules to representations of all maximal subgroups.

$SO(8) \supset Spin(7)$

$$8_v = 8$$

$$8_s = 1 + 7$$

$$8_c = 8$$

$$28 = 7 + 21$$

$$35_s = 1 + 7 + 27$$

$$35_c = 35$$

$SO(8) \supset SO(7)$

$$8_v = 1 + 7$$

$$8_s = 8$$

$$8_c = 8$$

$$28 = 7 + 21$$

$$35_i = 35 \quad ; \quad i = s, c .$$

(For SO(8) representations, the v, s & c indexing the $\underline{8}$'s denote vector, spinor and second spinor; s & c indexing the $\underline{35}$'s denote self-dual and anti-self-dual 4-forms; the third $\underline{35}$: 35_v is a symmetric traceless two-tensor.)

$SO(8) \supset SU(3)/Z_3$

$$8_i = 8 \quad ; \quad i = v, c, s .$$

$$28 = 8 + 10 + 10$$

$$35_i = 35$$

$SO(8) \supset SU(4) \otimes U(1)/Z_4$

$$8_v = 4_1 + \bar{4}_{-1}$$

$$8_s = 1_2 + 1_{-2} + 6_0$$

$$8_c = 4_{-1} + \bar{4}_1$$

$$28 = 1_0 + 6_2 + 6_{-2} + 15_0$$

$$35_s = 1_4 + 1_{-4} + 6_2 + 6_{-2} + 20_0 + 1_0$$

$$35_c = 15_0 + 10_2 + \overline{10}_{-2}$$

$SO(8) \supset SO(6) \otimes SO(2)$

$$8_v = 1_2 + 1_{-2} + 6_0$$

$$8_s = 4_1 + \bar{4}_{-1}$$

$$28 = 1_0 + 6_2 + 6_{-2} + 15_0$$

$$35_s = 15_0 + 10_2 + \overline{10}_{-2}$$

$SO(8) \supset Sp(4) \otimes SU(2)/Z_2$

$$8_v = (4,\ 2)$$

$$8_s = (5,\ 1) + (1,\ 3)$$

$$8_c = (4,\ 2)$$

$$28 = (1,\ 3) + (10,\ 1) + (5,\ 3)$$

$$35_s = (14,\ 1) + (5,\ 3) + (1,\ 5) + (1,\ 1)$$

$$35_c = (10,\ 3) + (5,\ 1)$$

$SO(8) \supset SO(5) \otimes SO(3)$

$$8_v = (5,\ 1) + (1,\ 3)$$

$$8_i = (4,\ 2) \quad , \quad i = s, c\ .$$

$$28 = (1,\ 3) + (10,\ 1) + (5,\ 3)$$

$$35_i = (10,\ 3) + (5,\ 1) \quad ; \quad i = s, c\ .$$

$SO(8) \supset SO(4) \otimes SO(4)$

$$8_v = (2,\ 2;\ 1,\ 1) + (1,\ 1;\ 2,\ 2)$$

$$8_s = (1,\ 2;\ 1,\ 2) + (2,\ 1;\ 2,\ 1)$$

$$8_c = (1,\ 2;\ 2,\ 1) + (2,\ 1;\ 1,\ 2)$$

$$28 = (1,\ 1;\ 1,3) + (1,1;\ 3,1) + (1,3;\ 1,1) + (3,1;\ 1,1) + (2,2;\ 2,2)$$

$$35_s = (1,1;1,1) + (2,2;2,2) + (3,1;3,1) + (1,3;1,3)$$

$$35_c = (1,1;1,1) + (2,2;2,2) + (3,1;1,3) + (1,3;3,1)$$

SO(7) ⊃ SO(6)

$7 = 1 + 6$

$8 = 4 + 4$

$21 = 6 + 15$

$35 = 15 + 4 + 6 + 6 + 4$

SO(7) ⊃ SO(5) ⊗ SO(2)

$7 = 1_2 + 1_{-2} + 5_0$

$8 = 4_1 + 4_{-1}$

$21 = 1_0 + 5_2 + 5_{-2} + 10_0$

$35 = 10_2 + 10_{-2} + 10_0 + 5_0$

SO(7) ⊃ G_2

$7 = 7$

$8 = 1 + 7$

$21 = 7 + 14$

$35 = 1 + 27 + 7$

SO(7) ⊃ SO(4) ⊗ SO(3)

$7 = (1,1;3) + (2,2;1)$

$8 = (1,2;2) + (2,1;2)$

$21 = (1,1;3) + (1,3;1) + (3,1;1) + (2,2;3)$

$35 = (1,1;1) + (1,3;3) + (3,1;3) + (2,2;3) + (2,2;1)$

SO(6) ⊃ SU(3) ⊗ U(1)/Z_3

$4 = 1_3 + 3_{-1}$

$6 = 3_2 + \bar{3}_{-2}$

$15 = 1_0 + 3_{-4} + \bar{3}_4 + 8_0$

SO(6) ⊃ SO(4) ⊗ SO(2)

$4 = (2,1)_1 + (1,2)_{-1}$

$6 = (1,1)_2 + (1,1)_{-2} + (2,2)_0$

$15 = (1,1)_0 + (3,1)_0 + (1,3)_0 + (2,2)_2 + (2,2)_{-2}$

SO(6) ⊃ Sp(4)

$4 = 4$

$6 = 1 + 5$

$15 = 5 + 10$

SO(6) ⊃ SO(4)

$4 = (2,2)$

$6 = (1,3) + (3,1)$

$15 = (1,3) + (3,1) + (3,3)$

SO(5) ⊃ SO(4)

$4 = (2,1) + (1,2)$

$5 = (1,1) + (2,2)$

SO(5) ⊃ SO(3)

$4 = 4$

$5 = 5$

SO(5) ⊃ SO(3) ⊗ SO(2)

$4 = 2_1 + 2_{-1}$

$5 = 1_2 + 1_{-2} + 3_0$

ii) Some useful properties of the octonions. [100]

The real octonion algebra is an 8-dimensional division algebra whose elements may be decomposed:

$$O = a_0 e_0 + \sum_{a=1}^{7} a_a e_a ,$$

where a_0 and a_a are real numbers, e_0 is the identity element and e_a are the seven imaginary units obeying the multiplication rule

$$e_a e_b = -\delta_{ab} + c_{abc} e_c ,$$

where c_{abc} is totally antisymmetric with nonvanishing components given by e.g. (6-9').

The associator (O_1, O_2, O_3) of any three octonions:

$$(O_1 , O_2 , O_3) \equiv (O_1 O_2) O_3 - O_1 (O_2 O_3)$$

is fully antisymmetric, i.e.

$$(O_1 , O_2 , O_3) = (O_3 , O_1 , O_2) = -(O_2 , O_1 , O_3) ;$$

and this implies, e.g. from $(e_i , e_j , e_j) = 0$ (sum over j), that

$$c_{irs} c_{jrs} = 6 \delta_{ij}$$

It also implies the Moufang identity:

$$(O_1 O_2)(O_3 O_1) = O_1 (O_2 O_3) O_1 ,$$

which implies

$$c_{ris} c_{sjt} c_{tkr} = 3 c_{ijk} .$$

The associator of any three imaginary units yields:

$$(e_i , e_j , e_k) = 2 \varphi_{ijkr} e_r = 2 c^{\ell}{}_{[ij} c_{k]}{}^{\ell}{}_{r} e_r ,$$

where the 4-form φ_{ijkr} is given by

$$\varphi_{ijkr} = \frac{1}{3!} \epsilon_{ijkr\ell mn} c_{\ell mn} = (\delta_{jk} \delta_{ir} - \delta_{ik} \delta_{jr}) + c_{ijs} c_{krs} .$$

The latter relation implies:

$$c_{ijk} = -\frac{1}{4!} \epsilon_{ijk\ell mnr} \varphi_{\ell mnr} = -\frac{1}{4!} \epsilon_{ijk\ell mnr} c_{\ell ms} c_{nrs} .$$

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Nuclear Physics: NP ; Physics Letters: PL ; Physical Review: PR ;

Physical Review Letters: PRL ; Commun. Math. Phys.: CMP .

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